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## RESEARCH STATEMENT

I am seeking a job with a strong emphasis on teaching, yet I know that an ongoing research program is a vital part of a professional mathematician's life. In this document I will discuss the research I was involved in as a graduate student, the project on which I am currently working, and a question that I may pursue in the future. All of these projects fall under the general heading of algebraic topology.

## GRADUATE SCHOOL

Let  $G$  denote a finite  $p$ -group and let  $H^*(G)$  denote  $H^*(G; \mathbb{F}_p)$ , the mod  $p$ -cohomology ring of  $G$ . Let  $\text{Out}(G)$  be the group of outer automorphisms of  $G$  and let  $e$  be an idempotent in  $\mathbb{F}_p[\text{Out}(G)]$ . Finally, let  $r_p(G)$  denote the rank of the largest elementary abelian  $p$ -subgroup of  $G$ ; this is called the  $p$ -rank of  $G$ . In this setting,  $eH^*(G)$  is finitely generated over the Noetherian ring  $H^*(G)^{\text{Out}(G)}$ , meaning that it has a Krull dimension, which we denote by  $\dim(eH^*(G))$ . In [13], D. Quillen showed that  $\dim(H^*(G)) = r_p(G)$ , thus establishing an upper bound for  $\dim(eH^*(G))$ . The following question was raised by J. Martino and S. Priddy in [9].

**Question 1** *Is  $\dim(eH^*(G)) = r_p(G)$  for every nonzero idempotent  $e \in \mathbb{F}_p[\text{Out}(G)]$ ?*

Since there is an association between idempotents and irreducible  $\mathbb{F}_p[\text{Out}(G)]$  modules, this question can be understood to concern the frequency of the occurrence of certain modules as composition factors in  $H^*(G)$ .

When  $G$  is a  $p$ -group,  $H^*(G)$  is an unstable module over the mod  $p$ -Steenrod algebra  $\mathcal{A}_p$ . In [14] L. Schwartz introduced the nilpotent filtration of the category  $\mathcal{U}$  of unstable modules over the Steenrod algebra. This filtration consists of a collection of nested, localizing subcategories  $\mathcal{N}il_n$ , so that we can discuss the  $\mathcal{N}il_n$ -localization and the  $\mathcal{N}il_n$ -closure of any module  $M \in \mathcal{U}$ . For any such module, we obtain a sequence of submodules

$$\cdots \subseteq \text{nil}_3(M) \subseteq \text{nil}_2(M) \subseteq \text{nil}_1(M) \subseteq M,$$

where each subquotient  $\text{nil}_s(M)/\text{nil}_{s+1}(M)$  is the  $s$ -fold suspension of a reduced unstable module. We will use the notation  $R_s(M)$  to denote this reduced module, so that

$$\text{nil}_s(M)/\text{nil}_{s+1}(M) \cong \Sigma^s R_s(M).$$

Let  $\overline{R}_s(M)$  denote the  $\mathcal{N}il_1$ -closure of  $R_s(M)$ .

Quillen proved the upper bound on  $\dim(eH^*(G))$  by studying the map

$$H^*(G) \longrightarrow \lim_{E \in \mathcal{A}(G)} H^*(E),$$

where  $\mathcal{A}(G)$  is the category of elementary abelian  $p$ -subgroups of  $G$ . In our notation, Quillen's map is the  $\mathcal{N}il_1$ -localization map and  $\lim_{E \in \mathcal{A}(G)} H^*(E)$  is the same thing as  $\overline{R}_0(H^*(G))$ . My work involved a detailed look at  $\overline{R}_i(H^*(G))$  for all  $i$ .

The following three propositions relate directly to Question 1. The first of these propositions shows that one may be able to answer this question through an examination of the building blocks  $\overline{R}_s(H^*(G))$ .

**Proposition 2** *With  $\overline{R}_s(H^*(G))$  defined as above,  $\dim(eH^*(G)) = \max_s \{\dim(e\overline{R}_s(H^*(G)))\}$ .*

The most desirable progress toward answering Question 1 would be a theorem which answers the question for all groups of a certain type. Optimal results in group cohomology usually have this form – conclusions about  $H^*(G)$  are drawn purely from group theoretic properties of  $G$ . I proved two such propositions. We need one definition in order to state the first of these results: a group  $G$  is called a  *$p$ -central group* if all its elements of order  $p$  are central.

**Proposition 3** *Let  $G$  be a  $p$ -central group. Then we have  $\dim(e\overline{R}_s(H^*(G))) = r_p(G)$  whenever  $e\overline{R}_s(H^*(G)) \neq 0$ .*

**Proposition 4** *If  $G$  contains an elementary abelian  $p$ -subgroup  $V$  which is self-centralizing, normal, and of maximal rank, then  $\dim(e\overline{R}_0(H^*(G))) = r_p(G)$  for all idempotents  $e \in \mathbb{F}_p[\text{Out}(G)]$ .*

Since calculations of the mod 2-cohomology of a large number of 2-groups are readily available (see either [3] or [5]), one can check the progress toward using the propositions above to answer Question 1 for 2-groups. Propositions 3 and 4, combined with some other established reasons (such as  $G$  abelian or  $\text{Out}(G)$  a 2-group), show that Question 1 has an affirmative answer for all but one of the 2-groups of order dividing 64.

This project has proved to be a fruitful area of research. Currently much more is known about Question 1; the most powerful results are contained in [8]. In particular, there is an affirmative answer to Question 1 for all odd primes and all 2-groups of order dividing 128.

## CURRENT RESEARCH

Randall Helmstutler (University of Mary Washington) and I both studied under Nicholas Kuhn in graduate school at the University of Virginia. Though our dissertation topics were unrelated, we spoke frequently with each other about our research, and thus the collaboration we began in the late spring of 2006 was a natural step. We are investigating a question in the area of category theory, and our work is still in the early stages. In a small way, as Dr. Helmstutler and I bring different areas of expertise to bear on the problem at hand, I grow to appreciate the goals of a liberal arts education even more.

Our project revolves around the question of when two functor categories are equivalent. That is, given a fixed category  $\mathcal{C}$  and small categories  $\mathcal{A}$  and  $\mathcal{B}$ , under what conditions will the categories

$[\mathcal{A}, \mathcal{C}]$  and  $[\mathcal{B}, \mathcal{C}]$  be equivalent? One can think of functor categories as a generalization of  $R$ -modules, so this question is in the same spirit as the question of the Morita equivalence of two rings.

Newell and Palmquist gave an early example of this type of exploration in [10] and [11]. More recent work has been done, with abelian or stable model categories in mind, in [12], [15], and [6]. We are trying to unify the existing results in a straightforward way.

We say that a category is *idempotent complete* if every idempotent in the category splits. By looking at [7, Lemma 1.4.8], it is clear that such categories are worth studying, since abelian categories as well as stable homotopy categories can be described in this way. Keeping in mind the previous work done in this area, we are led to the following question.

**Question 5** *Given small categories  $\mathcal{A}$  and  $\mathcal{B}$  and an idempotent complete category  $\mathcal{C}$ , under what conditions are the functor categories  $[\mathcal{A}, \mathcal{C}]$  and  $[\mathcal{B}, \mathcal{C}]$  equivalent?*

Thus, an answer to Question 5 will include the work in both abelian ([12]) and stable model categories ([6]) as corollaries.

As we work on this problem, two primary examples serve as our motivation. The first is discussed in detail in both [12] and [6]. Let  $\mathcal{E}$  denote the category which has as objects the sets  $\mathbf{n} = \{1, 2, \dots, n\}$ ; the morphisms in this category are the surjections. Let  $\Gamma$  denote the category which has as objects the based sets  $\mathbf{n}_+ = \{0, 1, 2, \dots, n\}$  where 0 serves as the basepoint; the morphisms here are based maps. In [12], Pirashvili proved that  $[\Gamma^{\text{op}}, \mathcal{C}]$  and  $[\mathcal{E}^{\text{op}}, \mathcal{C}]$  are equivalent when  $\mathcal{C}$  is an abelian category.

The second motivating example comes from the construction of the idempotent completion of a category. The idempotent completion  $\bar{\mathcal{A}}$  of a category  $\mathcal{A}$  is simply the smallest category containing  $\mathcal{A}$  in which all of the idempotents from  $\mathcal{A}$  split. It is a well-known fact (see [2, Proposition 1.3]) that the functor categories  $[\mathcal{A}, \mathcal{C}]$  and  $[\bar{\mathcal{A}}, \mathcal{C}]$  are equivalent for any idempotent complete category  $\mathcal{C}$ .

In these examples, we note that there is a nice embedding functor between the two domain categories in question. We are currently exploring the necessary properties of such a functor (and, necessarily, of the categories  $\mathcal{A}$  and  $\mathcal{B}$ ) to answer Question 5.

## FUTURE RESEARCH PLANS

While completing my dissertation work, I spent a good deal of time examining the mod 2-cohomology calculations of 2-groups found in [3]. I was studying the relationship between the 2-rank of a 2-group and the structure of its cohomology ring and in the process, I noticed something which may serve as a starting point for my next research project.

If  $H$  is a subgroup of a  $p$ -group  $G$ , then the inclusion of  $H$  into  $G$  gives rise to a restriction map in group cohomology,  $H^*(G) \rightarrow H^*(H)$ . When asking how these cohomology rings differ, it is natural to examine the kernel of this map. We say that an element  $x$  of  $H^*(G)$  is an *essential*

*element* if it is in the kernel of the restriction map for every subgroup  $H$  of  $G$ . The collection of all such elements forms the *essential ideal* of  $H^*(G)$ .

In [1], A. Adem and D. Karagueuzian proved a theorem characterizing  $p$ -central groups as groups with Cohen-Macaulay cohomology and nonzero essential cohomology. While examining the mod 2-cohomology calculations of 2-groups of order dividing 64, I noticed that every group which has maximal elementary abelian 2-subgroups of different ranks has no essential cohomology. Some facts relating to this structure are already known (see [13]), but this particular connection has not yet been explored. I believe that the depth of the cohomology rings may end up playing a key role in this problem, so among other sources I plan to consult [4] as I investigate this behavior in my future work.

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