The Nilpotent Filtration in Group Cohomology

Ryan Sterling Higginbottom
Charlottesville, Virginia

B.S., Bucknell University, 1999
M.S., University of Virginia, 2001

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
August 2005

Abstract

Let P be a finite p-group and let e be an idempotent in $\mathbb{F}_p[\mathrm{Out}(P)]$. In this dissertation we explore the Krull dimension of $eH^*(P;\mathbb{F}_p)$. It is known that this dimension cannot exceed the largest rank of an elementary abelian p-subgroup of P. We investigate conditions on P which ensure that $\dim(eH^*(P))$ is maximal.

The nilpotent filtration of the category of unstable modules over the Steenrod algebra plays a key role in the solutions we present. In particular, the dimension of a module depends only on the size of the subquotients in its nilpotent filtration. We also rely on the descriptions of the localization of $\mathcal{U}$ with respect to the categories $\mathcal{N}il_n$ given by H. W. Henn, J. Lannes, and L. Schwartz in [19] and [20].

Our main results come in the form of two separate group theoretic criteria. For a group P, $\dim(eH^*(P))$ is maximal if:

- P has an elementary abelian p-subgroup of maximal rank which is both normal in P and self-centralizing; or
- all elements of order p are central.

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Acknowledgements

My advisor, Nicholas Kuhn, has generously shared his time, energy, ideas, and advice with me over the past years, and I owe him a debt of gratitude. Additionally, I am grateful to Gregory Arone, Brian Parshall, and Daniel Keenan for serving on my dissertation committee.

My family and friends have also been tremendously supportive during this endeavor. Specifically, I wish to thank my parents, who have been constant in their interest, encouragement and love.

Finally, my deepest love and appreciation are reserved for my wife, Leanne. She has been patient, caring, encouraging, and helpful in more ways than I could ever recount.

"Here I raise my Ebenezer,
Hither by Thy help I'm come."

Soli Deo Gloria

Introduction

Let P be a finite p-group and let $H^*(P)$ denote $H^*(P;\mathbb{F}_p)$, the mod p-cohomology of P. It is well known that $H^*(P)$ is a module over $\mathbb{F}_p[\mathrm{Out}(P)]$, where $\mathrm{Out}(P)$ is the outer automorphism group of P. If e is a nonzero idempotent element of $\mathbb{F}_p[\mathrm{Out}(P)]$, one can inquire about the size of $eH^*(P)$ as a graded vector space over $\mathbb{F}_p$. In 1972, D. Quillen ([30]) showed that the Krull dimension of $H^*(P)$ is equal to the rank of the largest elementary abelian p-subgroup of P. We will denote this rank by $\mathrm{r}_p(P)$. In this dissertation we consider the following question.

Question 1 *Does the equality* $\dim(eH^*(P)) = \mathrm{r}_p(P)$ *always hold?*

Throughout our work we exploit the structure of $H^*(P)$ as an unstable module over the Steenrod algebra. The category of all such objects is denoted $\mathcal{U}$. For each $n \geq 1$, there is a notion of *n-nilpotence*, and the full subcategory of $\mathcal{U}$ consisting of n-nilpotent modules is denoted $\mathcal{N}il_n$. These subcategories are nested, forming a decreasing filtration of $\mathcal{U}$,

$$\cdots \subseteq \mathcal{N}il_3 \subseteq \mathcal{N}il_2 \subseteq \mathcal{N}il_1 = \mathcal{N}il \subseteq \mathcal{U},$$

known as the *nilpotent filtration*. If $\operatorname{nil}_n M$ denotes the largest submodule of M contained in $\mathcal{N}il_n$, we also have a filtration of each unstable module M:

$$\cdots \subseteq \operatorname{nil}_3 M \subseteq \operatorname{nil}_2 M \subseteq \operatorname{nil}_1 M \subseteq M.$$

The subquotients in this filtration have a particularly nice description. An unstable module is called *reduced* if it contains no nontrivial suspensions. For each $s \geq 0$, we are able to define a reduced module $R_s(M)$ by the formula

$$\Sigma^s R_s(M) \cong \operatorname{nil}_s M / \operatorname{nil}_{s+1} M.$$

The existence of the modules $R_s(M)$ is one of the defining characteristics of the nilpotent filtration, and we will use these modules to gather information about M.

In order to calculate the dimension of an unstable module, it will be necessary to focus on the subcategory of $\mathcal{U}$ denoted $K_{fg} - \mathcal{U}$. If M is a module in $K_{fg} - \mathcal{U}$, the nilpotent filtration of M is finite, meaning that there are a finite number of reduced modules $R_s(M)$.

Every unstable reduced module embeds in a terminal reduced module called its $\mathcal{N}il$*-closure*. Our approach to Question 1 involves drawing conclusions about the dimension of a module M from the dimension of the $\mathcal{N}il$-closure of the modules $R_s(M)$. We denote the $\mathcal{N}il$-closure of $R_s(M)$ by $\overline{R}_s(M)$. If M is a module in $K_{fg} - \mathcal{U}$, then $\dim(\overline{R}_s(M))$ is defined for every s, and we prove the following theorem in Section 3.4. We will apply this theorem when $M = eH^*(P)$.

Theorem *If* $M \in K_{fg} - \mathcal{U}$, *then* $\dim(M) = \max\{\dim(\overline{R}_s(M))\}$.

The category $\mathcal{A}(P)$ consists of the elementary abelian p-subgroups of P. Our first result, which appears in Section 4.3, requires only the use of $\overline{R}_0(H^*(P))$ in the above theorem, which is familiar as Quillen's inverse limit $\varprojlim_{\mathcal{A}(P)} H^*(V)$.

H. W. Henn, J. Lannes, and L. Schwartz studied the localization of $\mathcal{U}$ with respect to $\mathcal{N}il$ in [19], and they showed that $\mathcal{U}/\mathcal{N}il$ is equivalent to a certain category of functors. Let W denote an elementary abelian p-group and consider the set $\mathrm{Hom}(W, P)$. Via conjugation, there is an action of P on this set, and we denote the set of orbits of this action by $\mathrm{Rep}(W, P)$. In the equivalence of Henn, Lannes, and Schwartz, the module $\overline{R}_0(H^*(P))$ corresponds to the functor that sends W to $\mathbb{F}_p^{\mathrm{Rep}(W,P)}$.

It is easy to see that $\mathrm{Rep}(W, P)$ is an $\mathrm{Out}(P)$-set: if $[\alpha] \in \mathrm{Rep}(W, P)$ and $[f] \in \mathrm{Out}(P)$, then $[f] \cdot [\alpha] := [f \circ \alpha]$. If V is an elementary abelian p-subgroup of P, and ϕ is a map in $\mathrm{Hom}(W, P)$ with $\phi(W) = V$, we define $\mathrm{Out}(P)_V$ by the formula

$$\mathrm{Out}(P)_V := \{[f] \in \mathrm{Out}(P) \mid [f] \cdot [\phi] = [\phi]\}.$$

(This is independent of the choice of W and ϕ.) If $\mathrm{Out}(P)_V$ is a p-group, then it is easy to check that every simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module occurs as a composition factor in $\mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_V]$. In turn, we show that this implies $\dim(eH^*(P)) \geq \mathrm{rk}(V)$ for every nonzero idempotent e. This leads to the following result, in which we give group theoretic conditions which ensure that $\mathrm{Out}(P)_V$ is a p-group. (This initial answer to Question 1 is a slightly abbreviated version of Corollary 4.19.)

Theorem *If P contains an elementary abelian p-subgroup of maximal rank that is self-centralizing and normal, then* $\dim(eH^*(P)) = \mathrm{r}_p(P)$.

In Chapter 5 we turn to a description of the modules $\overline{R}_n(H^*(P))$, and for this we require a second paper of Henn, Lannes, and Schwartz. In [20] these authors examined localization of $\mathcal{U}$ away from $\mathcal{N}il_n$ for all $n \geq 1$. If $L_n : \mathcal{U} \to \mathcal{U}$ is the localization functor corresponding to $\mathcal{N}il_n$, then according to [20, Theorem I.5.5] one can calculate $L_n(H^*(P))$ by the following limit over $\mathcal{A}(P)_\sharp$:

$$\lim_{E \xrightarrow{\alpha} E'} \left[\mathrm{Eq} : \left\{ H^*(E) \otimes \Big(H^*(C_P(E')) \Big)^{<n} \overset{\mu(\alpha)}{\underset{\nu(\alpha)}{\rightrightarrows}} H^*(E) \otimes \Big(H^*(E \times C_P(E')) \Big)^{<n} \right\} \right]. \quad (*)$$

Let C denote the largest central elementary abelian p-subgroup of P, and let $\mathcal{A}_C(P)$ denote the full subcategory of $\mathcal{A}(P)$ whose objects are those in $\mathcal{A}(P)$ which contain C. In Chapter 6 we prove that there is an easier way to calculate $L_n(H^*(P))$.

Theorem *The limit in the formula* $(*)$ *to calculate* $L_n(H^*(P))$ *can be taken over* $\mathcal{A}_C(P)_\sharp$.

After we prove that the calculation of $L_n(H^*(P))$ can be taken over $\mathcal{A}_C(P)_\sharp$ instead of $\mathcal{A}(P)_\sharp$, we show that the same holds for $\overline{R}_n(H^*(P))$. For certain groups, this reduction in the formula for $\overline{R}_n(H^*(P))$ provides a second answer to Question 1. If a group has the property that every element of order p is central, we say that it is a *p-central group*. In this situation, there is a unique maximal elementary abelian p-subgroup. Using this structure, we prove the following theorem, which appears later as Theorem 7.4.

Theorem *If* P *is a p-central group, then* $\dim(eH^*(P)) = \mathrm{r}_p(P)$.

This theorem was essentially proved by J. Martino and S. Priddy in [26]. In this paper the authors posed a version of Question 1; we will explain the connections between [26] and the question we consider in Section 1.4.

In addition to the calculations we have described thus far, we include several examples in Chapter 8. Since p-central groups can be viewed as central extensions, we will use the Lyndon-Hochshield-Serre spectral sequence to calculate the cohomology of a p-central group of order 32. Additionally, we present explicit descriptions of the modules $R_n(H^*(P))$ and $\overline{R}_n(H^*(P))$ in the cases of the quaternion group of order 8 and the previously mentioned group of order 32. Finally, since calculations of the mod 2-cohomology of a large number of 2-groups are readily available, one can check the progress toward answering Question 1 for 2-groups. A careful inspection reveals only a single 2-group of order dividing 64 for which Question 1 is not settled. We include a table at the end of Chapter 8 containing this information.

Chapter 1

The question of Martino-Priddy

It is our goal in the present chapter to describe the motivation for the problem we have undertaken in this dissertation. In Sections 1.1-1.2 we survey the background to the problem as it was first posed and supply some of the information necessary to understand it in context. We give a precise statement of the original question in Section 1.3 and detail the situations in which the answer is already known. Section 1.4 discusses the specific restatement of this problem which occupies the rest of our work.

Throughout this chapter, we present a broad overview rather than a detailed study of the theory; references are provided at each stage for the interested reader. The appropriate technical considerations will be handled in subsequent chapters.

1.1 Idempotents and splittings of spectra

This section begins with a discussion of objects known as spectra and a description of their smallest irreducible subobjects. Both [25] and [27] are good resources to consult for background material.

A stable splitting of a spectrum X as $Y \vee Z$ is an isomorphism in the stable homotopy category $X \to Y \vee Z$. The spectra Y and Z are called *stable summands* of X if such a splitting exists.

Definition 1.1 A spectrum X is called *indecomposable* if there exists no non-trivial splitting $X \simeq X_1 \vee X_2$. A splitting $X \simeq X_1 \vee \cdots \vee X_n$ is called *complete* if X_i is indecomposable for $i = 1, \ldots, n$.

A splitting of X as $Y \vee Z$ gives a correspondence between summands of X and idempotent maps in the ring $\{X, X\}$. It is fairly simple to write the formula for the idempotent in $\{X, X\}$ corresponding to Y:

$$e_Y : X \simeq Y \vee Z \xrightarrow{\pi_Y} Y \xrightarrow{i_Y} Y \vee Z \simeq X.$$

Conversely, any idempotent $e \in \{X, X\}$ gives a splitting of X as $X \simeq eX \vee (1-e)X$ by means of the mapping telescope construction. There is a one-to-one correspondence between stable homotopy types of stable summands of a spectrum X and equivalence classes of idempotents in $\{X, X\}$.

Definition 1.2 Two idempotents e_1 and e_2 in a ring R are said to be *orthogonal* if $e_1 e_2 = e_2 e_1 = 0$. Additionally, we say that an idempotent $e \in R$ is *primitive* if it cannot be written as $e = e_1 + e_2$, where e_1 and e_2 are orthogonal idempotents.

An idempotent e in $\{X, X\}$ is primitive if and only if eX is indecomposable. This means that a primitive orthogonal decomposition of $1 \in \{X, X\}$ gives a complete splitting of X:

$$\left\{1 = \sum_{i=1}^{n} e_i\right\} \longleftrightarrow \left\{X \simeq \bigvee_{i=1}^{n} e_i X\right\}.$$

Having briefly surveyed splittings of spectra in general, we will now discuss the suspension spectrum of the classifying space of a finite p-group P. We use BP to denote the classifying space of P and $\Sigma^\infty BP$ will denote its suspension spectrum. We now turn our attention to specific summands that occur within a decomposition of $\Sigma^\infty BP$.

1.2 Summands of $\Sigma^\infty BP$

As we undertake an examination of stable splittings of BP for finite p-groups P, it becomes convenient to assign names to certain summands. The first type of summand we investigate is a *dominant summand* (see [29]).

If P is a finite p-group, let J be the ideal in $\{\Sigma^\infty BP, \Sigma^\infty BP\}$ generated by maps of the form

$$\Sigma^\infty BP \longrightarrow \Sigma^\infty BQ \longrightarrow \Sigma^\infty BP,$$

where Q is a proper subgroup of P.

Definition 1.3 An indecomposable summand X of $\Sigma^\infty BP$ is called *dominant* if the corresponding idempotent e_X does not belong to J. This is equivalent to saying that X is dominant if it does not have the homotopy type of a summand of $\Sigma^\infty BQ$ for $Q \lneq P$.

We denote the ring of p-adic integers by $\mathbb{Z}_p$ and the group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ of outer automorphisms of a group G by $\mathrm{Out}(G)$. In 1985, G. Nishida ([29]) noted the following isomorphism:

$$\{\Sigma^\infty BP, \Sigma^\infty BP\}/J \cong \mathbb{Z}_p[\mathrm{Out}(P)].$$

We will use this equivalence to label two other summands of $\Sigma^\infty BP$. (See [26, Section 1].)

Definition 1.4 If $\sum_{j=1}^{t} \tilde{e}_j$ is a primitive orthogonal decomposition of $1 \in \mathbb{Z}_p[\mathrm{Out}(P)]$, we say that $\tilde{e}_j \Sigma^\infty BP$ is a *superdominant summand* of $\Sigma^\infty BP$.

This definition warrants a bit more discussion. Each $\tilde{e}_j$ can be decomposed further into primitive orthogonal idempotents within $\{\Sigma^\infty BP, \Sigma^\infty BP\}$; specifically, we have

$$\tilde{e}_j = e_{j0} + e_{j1} + \cdots + e_{jk},$$

where $e_{j0} \notin J$ and $e_{ji} \in J$ for $i = 1, \ldots, k$. From this description we see that each superdominant summand contains a unique dominant summand.

Definition 1.5 Since there is a correspondence between dominant summands of $\Sigma^\infty BP$ and irreducible representations of $\mathrm{Out}(P)$ over $\mathbb{F}_p$, we say that the dominant summand corresponding to the trivial representation is called the *principal dominant summand*.

Our problem, as it was originally posed, concerns certain properties of the dominant summands of $\Sigma^\infty BP$. We will describe these properties in the next section.

1.3 The dimension of dominant summands

The first question we will discuss was asked in an article by J. Martino and S. Priddy in 1992 ([26]). We require just a few more definitions before a precise statement can be given. Hereafter, we assume that all cohomology calculations are performed with $\mathbb{F}_p$-coefficients.

Definition 1.6 The *Poincaré series* of a summand X of $\Sigma^\infty BP$ is the formal power series

$$\mathrm{PS}(X,t) = \sum_{k\geq 0} \dim_{\mathbb{F}_p}(H^k X) t^k.$$

The *dimension* of a summand X is the order of the pole of $\mathrm{PS}(X,t)$ at $t = 1$. Our notation for this number is $\mathrm{d}(X)$.

If X is a summand of $\Sigma^\infty BP$, Martino and Priddy show that $\mathrm{d}(X)$ is well-defined in [26, Proposition 1.6].

Question 1.7 (Martino–Priddy) *Let P be a finite p-group and let X be a dominant summand of $\Sigma^\infty BP$. Is it always true that $\mathrm{d}(X) = \mathrm{r}_p(P)$?*

Martino and Priddy answer this question in the affirmative when X is the principal dominant summand. They also define an algebra of Dickson-like invariants $D(P)$ and give an affirmative answer when $H^*(P)$ is free over $D(P)$. In [30], Quillen proved that $\mathrm{d}(\Sigma^\infty BP) = \mathrm{r}_p(P)$, thereby establishing an upper bound on the dimension of a dominant summand. Question 1.7 is therefore asking if $\mathrm{d}(X)$ is always maximal.

We note that Question 1.7 has been answered for at least one broad category of groups. Using the work of J. Harris and N. Kuhn ([17]) and a calculation of S. Mitchell ([28]), one can see that $\mathrm{d}(X) = \mathrm{r}_p(P)$ when P is abelian.

We have explained the context of Question 1.7 as it originally appeared in [26]. However, the complexity of the ring $\{\Sigma^\infty BP, \Sigma^\infty BP\}$ leads us to present two altered versions of this question in the following section.

1.4 Adjusting the problem

We now shift the focus from dominant summands of $\Sigma^\infty BP$ to superdominant summands. By Definition 1.4 this means that we will now concentrate on idempotents in the ring $\mathbb{Z}_p[\mathrm{Out}(P)]$.

Question 1.8 *Let P be a finite p-group and let X be a superdominant summand of $\Sigma^\infty BP$. Is it always true that $\mathrm{d}(X) = \mathrm{r}_p(P)$?*

We now discuss the final items necessary to state our problem; we refer to [22, Chapter 1] for the following brief discussion. If $\sum_{i=1}^{t} \tilde{e}_i$ is a primitive orthogonal decomposition of $1 \in \mathbb{F}_p[\mathrm{Out}(P)]$, then there exists a primitive orthogonal decomposition $1 = \sum_{i=1}^{t} e_i$ in $\mathbb{Z}_p[\mathrm{Out}(P)]$ such that $\tilde{e}_i = e_i + p\mathbb{Z}_p[\mathrm{Out}(P)]$ for every i, and these e_i are unique up to conjugation by a unit of $\mathbb{Z}_p[\mathrm{Out}(P)]$. We therefore see a connection between superdominant summands of $\Sigma^\infty BP$ and primitive idempotents in $\mathbb{F}_p[\mathrm{Out}(P)]$. With this discussion, we can restate Question 1.8, and we highlight this as our dissertation problem.

Question 1.9 *Let S be a simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module and let e_S be the corresponding idempotent. Is it always true that $\mathrm{d}(e_S H^*(P)) = \mathrm{r}_p(P)$?*

Remark 1.10 We write $e_S H^*(P)$ instead of $H^*(e_S \Sigma^\infty BP)$ in the previous question because in what follows our techniques are largely group theoretic. Since e_S is an idempotent, these quantities are the same (see [4, p.190]).

We note that every simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module must appear as a composition factor in $H^*(P)$. Though proved multiple times in the literature, we have traced this back to [12].

We have phrased Question 1.9 in terms of simple $\mathbb{F}_p[\mathrm{Out}(P)]$-modules because it is possible to view this problem as an inquiry about the frequency with which such modules appear as composition factors of $H^*(P)$. Let S be a simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module and let a_n be the number of times S occurs as a composition factor in $H^n(P)$.

Then, define the formal power series $p_S(t)$ by the following formula:

$$p_S(t) = \sum_{k \geq 0} a_k t^k.$$

It is possible to show that $p_S(t)$ equals a constant multiple of $\mathrm{PS}(X, t)$, where X is the superdominant summand of $\Sigma^\infty BP$ corresponding to S.

The background in this chapter has been heavily influenced by the statement of Question 1.7 in [26]. We will provide a partial answer to Question 1.9, but our approach will be distinct from the methods of Martino and Priddy. We will exploit the structure of $H^*(P)$ as a module over the Steenrod algebra, and it is to this subject that we now turn.

Chapter 2

The nilpotent filtration of $\mathcal{U}$

The nilpotent filtration of the category $\mathcal{U}$ is a fundamental tool that we employ throughout this work. In this chapter, we will introduce this filtration and list some of its salient features.

In Section 2.1 we describe a collection of cohomology operations known as the Steenrod algebra. Section 2.2 discusses localization in an abelian category and Section 2.3 examines this construction within $\mathcal{U}$. We detail how neighboring subcategories in the nilpotent filtration fit together in Section 2.4, and we briefly present an example in Section 2.5.

2.1 The Steenrod algebra

One of the basic structures that undergirds all of the objects in this dissertation is the Steenrod algebra. Steenrod squares ($p = 2$) and powers (odd primes) arise as cohomology operations in the mod p-cohomology of topological spaces. Specifically, when $p = 2$, Sq^i is an operation which raises cohomological dimension by i:

$$\mathrm{Sq}^i : H^n(X; \mathbb{Z}/2) \to H^{n+i}(X; \mathbb{Z}/2).$$

When $p > 2$, the Bockstein β raises dimension by 1

$$\beta : H^n(X; \mathbb{Z}/p) \to H^{n+1}(X; \mathbb{Z}/p),$$

and P^i raises dimension by $2i(p-1)$:

$$\mathrm{P}^i : H^n(X; \mathbb{Z}/p) \to H^{n+2i(p-1)}(X; \mathbb{Z}/p).$$

These operations satisfy many desirable properties, including naturality with respect to maps $f : X \to Y$ and commutativity with suspension. They collectively form an algebra over which $H^*(X; \mathbb{Z}/p)$ is a graded module.

Though these operations arise in mod p-cohomology, the Steenrod algebra can be specified by a short list of axioms. The following definition and many other facts about the Steenrod algebra can be found in [33, Chapter 1].

Definition 2.1 The *Steenrod algebra* $\mathcal{A}_2$ is the graded $\mathbb{F}_2$-algebra generated by the elements Sq^i, $i \geq 0$, subject to the following relations:

$$\mathrm{Sq}^0 = 1,$$

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j=0}^{[\frac{a}{2}]} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j, \qquad \text{where } 0 < a < 2b. \tag{2.1}$$

For an odd prime p, $\mathcal{A}_p$ is the graded $\mathbb{F}_p$-algebra generated by the elements β and P^i, $i \geq 0$, subject to the following relations:

$$\beta^2 = 0,$$

$$\mathrm{P}^0 = 1,$$

$$\mathrm{P}^a \mathrm{P}^b = \sum_{j=0}^{[\frac{a}{p}]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \mathrm{P}^{a+b-j} \mathrm{P}^j, \qquad \text{where } 0 < a < pb, \tag{2.2}$$

$$\begin{aligned} \mathrm{P}^a \beta \mathrm{P}^b = &\sum_{j=0}^{[\frac{a}{p}]} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta \mathrm{P}^{a+b-j} \mathrm{P}^j \\ &+ \sum_{j=0}^{[\frac{a-1}{p}]} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} \mathrm{P}^{a+b-j} \beta \mathrm{P}^j, \\ &\text{where } 0 < a \leq pb. \end{aligned} \tag{2.3}$$

(The expressions in (2.1), (2.2) and (2.3) are known as the *Adem relations*.)

The original context of the mod p-cohomology of topological spaces motivates the following definition. We now write $\mathcal{A}$ for $\mathcal{A}_p$ when there is no concern for the prime involved. For the dimension of an element x of M we use the symbol $|x|$.

Definition 2.2 Let M be an $\mathcal{A}$-module. Then M is said to be *unstable* if

$\mathrm{Sq}^i x = 0$ whenever $i > |x|$ when $p = 2$, or

$$\beta^e \mathrm{P}^i x = 0 \text{ whenever } e + 2i > |x|, \ (e = 0, 1) \text{ when } p > 2.$$

We note that the mod p-cohomology of a space is unstable as a module over $\mathcal{A}_p$. Among other things, this condition implies that an unstable $\mathcal{A}$-module must be trivial in negative dimensions.

Notation We will denote the category of all unstable $\mathcal{A}$-modules by $\mathcal{U}$. The morphisms in $\mathcal{U}$ are $\mathcal{A}$-linear maps of degree zero. Hereafter, when referring to an object of $\mathcal{U}$, we will shorten the phrase "unstable module over the Steenrod algebra" to "unstable module."

We now define Steenrod operations that depend on the dimension of the elements on which they act. For $p = 2$, we use Sq_j to denote the following operation:

$$\begin{aligned} \mathrm{Sq}_j : M^k &\longrightarrow M^{2k-j} \\ x &\longmapsto \mathrm{Sq}^{k-j}\, x. \end{aligned}$$

For $p > 2$, we consider $e \in \{0, 1\}$ and define P_j by

$$\begin{aligned} \mathrm{P}_j : M^{2k+e} &\longrightarrow M^{2kp+e-2j(p-1)} \\ x &\longmapsto \mathrm{P}^{k-j} x. \end{aligned}$$

While unstable modules over the Steenrod algebra already exhibit abundant structure, objects such as the mod p-cohomology of a topological space are endowed with even more algebraic properties. These objects are graded, unital, $\mathbb{F}_p$-algebras, and there is a relationship between the algebra and $\mathcal{A}$-module structures. This is presented abstractly in the following definition.

Definition 2.3 We say that K is an *unstable* $\mathcal{A}$*-algebra* if it is a commutative, unital $\mathbb{F}_p$-algebra and an unstable $\mathcal{A}$-module, such that the following properties hold. (The first two properties must hold when $p = 2$; the last two must hold when $p > 2$.)

1. For any $x, y \in K$,
$$\mathrm{Sq}^k(xy) = \sum_{i+j=k} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y).$$

2. For any $x \in K$, $\mathrm{Sq}^{|x|}\,x = x^2$.

3. For any $x, y \in K$,
$$\mathrm{P}^k(xy) = \sum_{i+j=k} \mathrm{P}^i(x)\mathrm{P}^j(y), \text{ and}$$
$$\beta(xy) = (\beta x)y + (-1)^{|x|} x\beta y.$$

4. For any $x \in K$ where $|x| = 2k$, $\mathrm{P}^k x = x^p$.

(Properties 1 and 3 are collectively known as the *Cartan formula*.)

Notation We will denote the category of unstable algebras over the Steenrod algebra by $\mathcal{K}$. The morphisms in $\mathcal{K}$ are $\mathcal{A}$-linear algebra maps of degree zero. When referring to an object of $\mathcal{K}$ we will hereafter simply use the phrase "unstable algebra."

Since the underlying skeleton of an $\mathcal{A}$-module is a graded $\mathbb{F}_p$-vector space, there is an obvious notion of the suspension of such an object. If M is an unstable module, then we understand ΣM in the following way:

$$(\Sigma M)^n \cong M^{n-1}.$$

Furthermore, the action of $\mathcal{A}$ on ΣM is defined as follows: for any θ in $\mathcal{A}$ and any x in M,

$$\theta(\Sigma x) = (-1)^{|\theta|}\Sigma(\theta x).$$

It is easy to check that if M is an unstable module, so is ΣM; the converse, however, is false. With this understanding of suspensions, we make our final definition of this section.

Definition 2.4 An unstable module M is said to be *reduced* if it contains no non-trivial suspension as a submodule.

This section has provided a simple introduction to the category $\mathcal{U}$. We will soon discuss localizing subcategories of $\mathcal{U}$, but we must first acquire the necessary framework and terminology. This requires an excursion into an abstract discussion of localization.

2.2 Localization in an abelian category

Let $\mathcal{C}$ denote an abelian category and suppose that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$. We say that $\mathcal{B}$ is a *Serre class* in $\mathcal{C}$ when the following property holds: Let

$$0 \to C' \to C \to C'' \to 0$$

be a short exact sequence of objects and morphisms in $\mathcal{C}$. If any two of the objects are contained in $\mathcal{B}$, then so is the third. (This description of a Serre class is taken from [33, p.115]; the reader is referred to [15] as a good reference for all of the constructions in this section.)

A morphism $f : X \to Y$ in $\mathcal{C}$ is called a *$\mathcal{B}$-isomorphism* if both the kernel and cokernel of f are objects in $\mathcal{B}$. One often wishes to regard $\mathcal{B}$-isomorphisms as honest-to-goodness isomorphisms, and this is achieved through the construction of the quotient category $\mathcal{C}/\mathcal{B}$. The objects of the quotient category are the same as those of $\mathcal{C}$, and $\mathcal{C}/\mathcal{B}$ comes with the following universal property. If F is an additive functor from $\mathcal{C}$ to another abelian category $\mathcal{D}$ which is trivial on $\mathcal{B}$, then there exists a unique factorization of F through $\mathcal{C}/\mathcal{B}$,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {\scriptstyle T}\searrow \quad \nearrow{\scriptstyle F'} & \\ & \mathcal{C}/\mathcal{B}. & \end{array}$$

(Here $T : \mathcal{C} \to \mathcal{C}/\mathcal{B}$ is the canonical functor.) When $\mathcal{B}$ is a Serre class, T is exact and $\mathcal{C}/\mathcal{B}$ is again an abelian category.

Definition 2.5 We say that $\mathcal{B}$ is a *localizing subcategory* of $\mathcal{C}$ if T has a right adjoint; for the moment we will denote this $S : \mathcal{C}/\mathcal{B} \to \mathcal{C}$.

Remark 2.6 P. Gabriel provides this definition on [15, p.372] and shortly thereafter precisely states the conditions under which $\mathcal{B}$ is localizing (see [15, Corollary 1, p.375]). If $\mathcal{C}$ has injective envelopes, then a Serre class $\mathcal{B}$ is a localizing subcategory if every object of $\mathcal{C}$ contains a maximal subobject in $\mathcal{B}$. This criteria will apply to all of the categories we consider.

Definition 2.7 Suppose that $\mathcal{B}$ is a localizing subcategory of $\mathcal{C}$ and that C is an object in $\mathcal{C}$. We say that C is $\mathcal{B}$-*closed* if every $\mathcal{B}$-isomorphism $f : C_1 \to C_2$ gives rise to an isomorphism $f^* : \mathrm{Hom}_{\mathcal{C}}(C_2, C) \to \mathrm{Hom}_{\mathcal{C}}(C_1, C)$.

Several nice properties hold when $\mathcal{B}$ is a localizing subcategory of $\mathcal{C}$. If we define the functor $L : \mathcal{C} \to \mathcal{C}$ by $L := S \circ T$, then $L(C)$ is $\mathcal{B}$-closed for every object C in $\mathcal{C}$. It is also clear that the natural map $\eta_C : C \to L(C)$ is always a $\mathcal{B}$-isomorphism. We refer to $L(C)$ as the $\mathcal{B}$-*closure* of C and we say that the morphism η_C is *localization away from* $\mathcal{B}$. Finally, we see that η_C is initial among all maps from C to $\mathcal{B}$-closed objects of $\mathcal{C}$.

This brief discussion of localization now serves as the setting within which we shall work. In the following section we consider specific localizing subcategories of $\mathcal{U}$.

2.3 Localization in $\mathcal{U}$

In this section we will define a subcategory $\mathcal{N}il_n$ of $\mathcal{U}$ for every $n \geq 1$. We will also survey the work of H. W. Henn, J. Lannes, and L. Schwartz to describe localization in the case $n = 1$.

2.3.1 The categories $\mathcal{N}il_n$

We now introduce the categories with respect to which we will examine the localization of $\mathcal{U}$. Let V denote an elementary abelian p-group. The functor $T_V : \mathcal{U} \to \mathcal{U}$ is defined as left adjoint to tensoring a module with $H^*(V)$. (See [24] for information about T_V.) The following proposition appears in [21, Section 2].

Proposition 2.8 *If M is an unstable module, then the following conditions are equivalent.*

1. *The module M is the union of its submodules having a finite filtration whose subquotients are n-fold suspensions.*

2. *For all elementary abelian p-groups V, $T_V M$ is n-connected; that is, we have $(T_V M)^s = 0$ for all $s < n$.*

Definition 2.9 An unstable module is *n-nilpotent* if it satisfies either condition of Proposition 2.8.

There is also a characterization of n-nilpotent modules using Steenrod operations. An unstable module M is n-nilpotent if the operations Sq_j (or P_j) are locally nilpotent for all $0 \leq j < n$. That is, for each $x \in M$ and each j, $0 \leq j < n$, there must exist a number $l_{x,j}$ such that $\mathrm{Sq}_j^{l_{x,j}} x = 0$ (or $\mathrm{P}_j^{l_{x,j}} x = 0$).

The use of the word "nilpotent" here is justified by considering a specific case. If $p = 2$ and M is an unstable algebra, M is 1-nilpotent if the squaring map is locally nilpotent in the classical sense. The notion of n-nilpotence was introduced in [32], though the numbering has since been adjusted. Note that n-nilpotent here is $(n-1)$-nilpotent in the original terminology.

Define $\mathcal{N}il_n$ to be the full subcategory of $\mathcal{U}$ whose objects are the n-nilpotent unstable modules. (We will write $\mathcal{N}il$ for $\mathcal{N}il_1$.) We see that $\mathcal{N}il_n$ is the smallest localizing subcategory of $\mathcal{U}$ containing n-fold suspensions. Denote the localization functor corresponding to $\mathcal{N}il_n$ by $L_n : \mathcal{U} \to \mathcal{U}$ and the natural transformation $\mathrm{id}_{\mathcal{U}} \to L_n$ by λ_n. (We write L for L_1 and λ for λ_1 hereafter.) Then for each module M in $\mathcal{U}$, $L_n(M)$ is $\mathcal{N}il_n$-closed and $\lambda_{n,M} : M \to L_n(M)$ is a $\mathcal{N}il_n$-isomorphism. We will occasionally use the notation $\overline{M}$ instead of $L(M)$ to refer to the $\mathcal{N}il$-closure of M.

Since the subcategories $\mathcal{N}il_n$ are nested, we have a decreasing filtration of $\mathcal{U}$,

$$\cdots \subset \mathcal{N}il_3 \subset \mathcal{N}il_2 \subset \mathcal{N}il \subset \mathcal{U},$$

called the *nilpotent filtration*. Let $\operatorname{nil}_n : \mathcal{U} \to \mathcal{N}il_n$ be right adjoint to the inclusion of $\mathcal{N}il_n$ into $\mathcal{U}$. Then $\operatorname{nil}_n M$ is the largest n-nilpotent submodule of M. We thus realize the nilpotent filtration of $\mathcal{U}$ for any unstable module M in this way:

$$\cdots \subseteq \operatorname{nil}_3 M \subseteq \operatorname{nil}_2 M \subseteq \operatorname{nil}_1 M \subseteq M.$$

The following proposition shows that this filtration can be characterized by its subquotients.

Proposition 2.10 (Proposition 2.2 in [21]) *Let M be an unstable module.*

1. *For every s, $\operatorname{nil}_s M/\operatorname{nil}_{s+1} M$ is the s-fold suspension of a reduced unstable module.*

2. *If $\cdots \subseteq F_2M \subseteq F_1M \subseteq M$ is a filtration of M such that $F_sM/F_{s+1}M$ is the s-fold suspension of a reduced unstable module for all s, then $\operatorname{nil}_s M \subseteq F_sM$ for all s.*

Notation Following Proposition 2.10, for an unstable module M and an integer

$s \geq 0$, we define a reduced module $R_s(M)$ by the following relationship:

$$\Sigma^s R_s(M) = \operatorname{nil}_s M / \operatorname{nil}_{s+1} M. \tag{2.4}$$

We will write $\overline{R}_s(M)$ for the $\mathcal{N}il$-closure of $R_s(M)$.

In considering the localization of $\mathcal{U}$ with respect to the categories $\mathcal{N}il_n$, we are greatly indebted to H. W. Henn, J. Lannes, and L. Schwartz. They first examined this construction in [19], detailing localization of the categories $\mathcal{U}$ and $\mathcal{K}$ with respect to $\mathcal{N}il$. We will now briefly review their description of $\mathcal{U}/\mathcal{N}il$.

2.3.2 Localization with respect to $\mathcal{N}il$

Let $\mathcal{E}_\infty$ and $\mathcal{E}$ denote the categories of $\mathbb{F}_p$-vector spaces and finite dimensional $\mathbb{F}_p$-vector spaces, respectively. Denote by $\mathcal{F}$ the category of covariant functors from $\mathcal{E} \to \mathcal{E}_\infty$. We describe a functor f which assigns an object of $\mathcal{F}$ to an unstable module according to the following rule: if M is an unstable module and V is a finite dimensional $\mathbb{F}_p$-vector space, then $f(M)(V)$ is given by

$$f(M)(V) = \left(\operatorname{Hom}_{\mathcal{U}}(M, H^*(V))\right)'.$$

A careful examination of the modules $H^*(V)$ shows that if N is a nilpotent unstable module, then $\operatorname{Hom}_{\mathcal{U}}(N, H^*(V))$ is trivial for all V. The converse is true as well, meaning that $f(N)$ is the zero functor if and only if N is nilpotent. This is one of the first major steps in describing $\mathcal{N}il$-localization.

In [19, Section I.6] the authors discuss various classifications of objects in $\mathcal{F}$. In particular, they describe what it means for such a functor to be *analytic*. Let $\mathcal{F}_\omega$ denote the full subcategory of $\mathcal{F}$ whose objects are analytic functors. We learn that all values of f lie in $\mathcal{F}_\omega$; this is the final ingredient necessary to describe $\mathcal{U}/\mathcal{N}il$ in detail. The establishment of the equivalence $\mathcal{U}/\mathcal{N}il \cong \mathcal{F}_\omega$ occupies [19, Section I.7].

The strategy of this proof begins with the observation that f admits a right adjoint $m : \mathcal{F} \to \mathcal{U}$. If F is an object in $\mathcal{F}$, then $m(F)$ is given by

$$m(F) = \mathrm{Hom}_{\mathcal{F}}(H_*, F).$$

The fact that f and m are adjoint functors means that for every $M \in \mathcal{U}$ and every $F \in \mathcal{F}$ we have

$$\mathrm{Hom}_{\mathcal{F}}(f(M), F) \cong \mathrm{Hom}_{\mathcal{U}}(M, m(F)).$$

Since f vanishes on $\mathcal{N}il$, $m(F)$ is reduced for every F. Additionally, since f factors through $\mathcal{U}/\mathcal{N}il$ we see that $m(F)$ is $\mathcal{N}il$-closed. In Theorem 7.3 of [19], the authors show the existence of a natural isomorphism $m \circ f \cong L$.

We have only briefly surveyed the work of Henn, Lannes, and Schwartz to describe $\mathcal{U}/\mathcal{N}il$. These authors also studied localization with respect to $\mathcal{N}il_n$ for all n in [20].

These two articles contain much of the work upon which this dissertation depends. In particular, in proving the categorical equivalence between $\mathcal{U}/\mathcal{N}il_n$ and $\mathcal{F}_\omega^{<n}$ in [20], the authors provide us with the equipment necessary to examine the modules $\overline{R}_n(H^*(G))$. This investigation is what ultimately affords progress toward a partial answer to Question 1.9.

2.4 Neighboring subcategories

In this section we investigate the interaction between neighboring subcategories in the nilpotent filtration and thus provide a description of the modules $\overline{R}_s(M)$. Since we have a nesting $\mathcal{N}il_{n+1} \subseteq \mathcal{N}il_n$, a $\mathcal{N}il_n$-closed module is also $\mathcal{N}il_{n+1}$-closed. We therefore identify $L_{n+1}L_n(M)$ with $L_n(M)$ by means of $\lambda_{n+1,L_n(M)}$ and set $\tau_n := L_{n+1}\lambda_n$ to produce a natural transformation $\tau_n : L_{n+1} \to L_n$; the factorization $\tau_n \circ \lambda_{n+1} = \lambda_n$ is clear. For an unstable module M, we will now give a description of the kernel of $\tau_{n,M}$.

Proposition 2.11 *The kernel of the map $\tau_{n,M}$ is isomorphic to $\Sigma^n\overline{R}_n(M)$.*

Proof: The kernel of $\lambda_{n,M}$ is the largest submodule of M contained in $\mathcal{N}il_n$, so $\ker \lambda_{n,M} = \mathrm{nil}_n\, M$. For this proof we will abbreviate $\lambda_{n,M}$ by λ_n, the restriction of $\tau_{n,M}$ to $\lambda_{n+1}(M)$ by $\tilde{\tau}$, and the kernel of $\tilde{\tau}$ by $k_n(M)$. Consider how the localizations away from $\mathcal{N}il_n$ and $\mathcal{N}il_{n+1}$ fit together:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & k_n(M) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{nil}_{n+1} M & \xrightarrow{i} & M & \xrightarrow{\lambda_{n+1}} & \lambda_{n+1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \tilde{\tau}} & & \\
0 & \longrightarrow & \operatorname{nil}_n M & \xrightarrow{i} & M & \xrightarrow{\lambda_n} & \lambda_n(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Sigma^n R_n(M) & & 0\,. & & & &
\end{array}
$$

All of the squares in this diagram commute, and since we have exactness everywhere, we conclude $k_n(M) \cong \Sigma^n R_n(M)$ by the Snake Lemma.

Denote the cokernel of λ_n by $c_n(M)$ and assign the name $a_n(M)$ to the kernel of the map $c_{n+1}(M) \to c_n(M)$ induced by $\tau_{n,M}$. Then, we examine this commutative diagram, which is also exact everywhere:

$$
\begin{array}{ccccccccc}
 & & k_n(M) & & \ker \tau_{n,M} & & a_n(M) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \lambda_{n+1}(M) & \xrightarrow{i} & L_{n+1}(M) & \longrightarrow & c_{n+1}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{\tau}} & & \downarrow{\scriptstyle \tau_{n,M}} & & \downarrow & & \\
0 & \longrightarrow & \lambda_n(M) & \xrightarrow{i} & L_n(M) & \longrightarrow & c_n(M) & \longrightarrow & 0\,.
\end{array}
$$

We again employ the Snake Lemma and want to prove that the resulting map

$$\Sigma^n R_n(M) \cong k_n(M) \xrightarrow{\alpha} \ker \tau_{n,M}$$

is $\mathcal{N}il_{n+1}$-localization. To do this we must show that α is a $\mathcal{N}il_{n+1}$-isomorphism and that $\ker \tau_{n,M}$ is $\mathcal{N}il_{n+1}$-closed. The cokernel of α is isomorphic to a submodule of $a_n(M)$, which is itself contained in $c_{n+1}(M)$. Since $c_{n+1}(M) = \operatorname{coker} \lambda_{n+1}$, it is contained in $\mathcal{N}il_{n+1}$ by definition. Thus $\operatorname{coker} \alpha \in \mathcal{N}il_{n+1}$, and we conclude that α is a $\mathcal{N}il_{n+1}$-isomorphism since α is monic. Further, since $\ker \tau_{n,M}$ is the kernel of a map between $\mathcal{N}il_{n+1}$-closed modules, it is also $\mathcal{N}il_{n+1}$-closed.

We now show that the map $\rho : \Sigma^n R_n(M) \longrightarrow \Sigma^n \overline{R}_n(M)$ is also $\mathcal{N}il_{n+1}$-localization. The map $\lambda : R_n(M) \to \overline{R}_n(M)$ is $\mathcal{N}il$-localization by definition. Thus, since $\ker \lambda$ and $\operatorname{coker} \lambda$ are contained in $\mathcal{N}il$, $\Sigma^n \ker \lambda$ and $\Sigma^n \operatorname{coker} \lambda$ are contained in $\mathcal{N}il_{n+1}$. Finally, the fact that $\overline{R}_n(M)$ is $\mathcal{N}il$-closed means that $\Sigma^n \overline{R}_n(M)$ is $\mathcal{N}il_{n+1}$-closed. This last fact is non-trivial; it is a consequence of [9, Proposition 1.17].

Since we have demonstrated that both α and ρ are $\mathcal{N}il_{n+1}$-localization maps, and since $\Sigma^n R_n(M) \cong k_n(M)$, we conclude that $\Sigma^n \overline{R}_n(M) \cong \ker \tau_{n,M}$. □

We will give a more concrete description of the modules $\overline{R}_s(M)$ in Section 5.1 when we study the case $M = H^*(G)$. The relationship of $H^*(G)$ to localization away from $\mathcal{N}il_n$ has already been examined rather closely, and in the next section we present one of the earliest examples.

2.5 An example

While the localization away from $\mathcal{N}il$ was described in [19], the case of group cohomology had been studied earlier by D. Quillen in [30]. We briefly discuss his work here and relate it to our notation.

If P is a finite p-group, we define the category $\mathcal{A}(P)$. The objects of $\mathcal{A}(P)$ are the elementary abelian p-subgroups of P. The morphism sets in $\mathcal{A}(P)$ are defined as

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}(P)}(A,B) = \{f \in \mathrm{Hom}(A,B) \,|\, &\text{there exists } g \in P \text{ such that} \\ &f(a) = c_g(a) \text{ for all } a \in A\},\end{aligned}$$

where c_g denotes conjugation by the element g. This is the category that D. Quillen used in [30], and we will call this *Quillen's category*.

For each $V \in \mathcal{A}(P)$, the inclusion $V \hookrightarrow P$ gives rise to the restriction map in mod p-cohomology, $\mathrm{res}^P_V : H^*(P) \to H^*(V)$. Quillen showed that the map

$$q_P : H^*(P) \to \varprojlim_{\mathcal{A}(P)} H^*(V), \tag{2.5}$$

induced by the maps res^P_V, is a $\mathcal{N}il$-isomorphism.

Notation It will be convenient to assign notation to the inverse limit in (2.5), so we define $\mathcal{H}^*(P)$ by

$$\mathcal{H}^*(P) := \varprojlim_{\mathcal{A}(P)} H^*(V).$$

Henn, Lannes, and Schwartz note that Quillen's map q_G is localization away from $\mathcal{N}il$ ([20]). Using the terminology of Section 2.4, we say that for a p-group P, we have $\overline{R}_0(H^*(P)) = \mathcal{H}^*(P)$.

We will return to group cohomology in Chapter 4, but we must now acquire more of the background necessary to make this study fruitful. In the next chapter we will look carefully at the proper subcategory of $\mathcal{U}$ in which our investigations about dimension will occur.

Chapter 3

Dimension within $\mathcal{U}$

In Chapter 4 we will draw conclusions about the size of the modules $e_S H^*(P)$, and the measure of size we will use is called *dimension*. In Section 3.1 we develop two different ways to calculate the dimension of a module. We introduce the category $K_{fg} - \mathcal{U}$ in Section 3.2 and explain that this is the ideal setting in which to discuss dimension within $\mathcal{U}$. Section 3.3 shows that the modules $\overline{R}_s(M)$ inherit from M the characteristics necessary to perform dimension calculations. We will detail our strategy to use these modules in the calculation of the dimension of M in Section 3.4.

3.1 Background in dimension theory

In this section we will define the notion of the dimension of a module, and we will show that in some situations this number can be calculated in multiple ways. For the background in commutative algebra we refer the reader to [3, Chapter 11].

Let R be a nonzero ring. We say that the *dimension* of R, denoted $\dim(R)$, is the supremum of the lengths of proper chains of prime ideals in R; this is also called the *Krull dimension* of R. If M is a nonzero R-module, the *annihilator* of M, denoted $\operatorname{Ann}(M)$, is defined by

$$\operatorname{Ann}(M) := \{r \in R \mid rm = 0 \text{ for all } m \in M\}.$$

Definition 3.1 If M is an R-module, then the *dimension* of M is denoted by $\dim(M)$ and defined by the following equation:

$$\dim(M) := \dim(R/\operatorname{Ann}(M)).$$

We will now introduce another measure of the size of a module, and we will show that in certain circumstances, this notion coincides with dimension. If $V = \bigoplus_{i\geq 0} V_i$ is a graded vector space of finite type over a field k, the *Poincaré series* of V, denoted $\operatorname{PS}(V,t)$, is defined to be the following formal power series:

$$\operatorname{PS}(V,t) := \sum_{i\geq 0} (\dim_k V_i) t^i.$$

In certain situations, the Poincaré series of V has a rather nice form. This is shown by the following result of Hilbert and Serre. (The interested reader can find a proof in [3, p.117].)

Theorem 3.2 *Let $A = \oplus A_n$ be a noetherian graded ring. If M is a finitely generated graded module over A, then $\mathrm{PS}(M,t)$ is a rational function of the form*

$$\mathrm{PS}(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})},$$

where $f(t) \in \mathbb{Z}[t]$. (The numbers $k_1, \ldots, k_s$ are the degrees of the generators of A as an A_0-algebra.)

In Theorem 3.2, $\mathrm{PS}(M,t)$ measures the size of M as a graded vector space over the field A_0. We can now define a second measure of the size of a module.

Definition 3.3 If a module M satisfies the conditions of Theorem 3.2, then the *growth* of M, denoted $\mathrm{d}(M)$, is the order of the pole of $\mathrm{PS}(M,t)$ at $t = 1$.

Remark 3.4 Through the rest of this section, whenever we use the symbol $\mathrm{d}(M)$ we assume that the growth of M is well-defined.

Our present goal is to describe the conditions on a module under which the notions of growth and dimension coincide. These are set forth in the following proposition.

Proposition 3.5 *Let R be a commutative graded ring of finite type over a field k. If M is a finitely generated R-module, then $\mathrm{d}(M) = \dim(M)$.*

The proof of this proposition will occupy the rest of this section. We begin with a lemma.

Lemma 3.6 *Let M be a finitely generated R-module.*

(i) If M' and M'' are also finitely generated R-modules and

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence, then $\mathrm{d}(M) = \max\{\mathrm{d}(M'), \mathrm{d}(M'')\}$.

(ii) We have $\mathrm{d}(M) = \mathrm{d}(\Sigma M)$.

PROOF: The proof of (i) relies on the fact that vector space dimension is an *additive function* in the sense of [3, p.23]. This makes the Poincaré series additive over short exact sequences. If

$$\mathrm{PS}(M, t) = \mathrm{PS}(M', t) + \mathrm{PS}(M'', t),$$

we see that (i) holds by Theorem 3.2 and the definition of $\mathrm{d}(M)$. The proof of (ii) is immediate since the suspension functor multiplies the Poincaré series of a module by t. □

Remark 3.7 Lemma 3.6(i) is mentioned in [5, p.159], where the author notes two important consequences. First, if N is a direct sum of finitely many copies of M, then $\mathrm{d}(N) = \mathrm{d}(M)$. Additionally, since any finitely generated R-module M is a quotient of a direct sum of finitely many copies of R, $\mathrm{d}(M) \leq \mathrm{d}(R)$.

We require two more propositions before proving Proposition 3.5. We quote the first of these without proof.

Proposition 3.8 (Theorem 5.4.6(ii) in [5]) *If R is a finitely generated commutative graded ring of finite type over a field k, then* $\dim(R) = \mathrm{d}(R)$.

Proposition 3.9 *If M is a finitely generated R-module,* $\mathrm{d}(M) = \mathrm{d}(R/\mathrm{Ann}(M))$.

PROOF: By the definition of $\mathrm{Ann}(M)$, M is an $R/\mathrm{Ann}(M)$-module. Since M is finitely generated over R, it is also finitely generated over $R/\mathrm{Ann}(M)$. Remark 3.7 then states that $\mathrm{d}(M) \leq \mathrm{d}(R/\mathrm{Ann}(M))$.

To show the other inequality, let $\{x_1, \ldots, x_r\}$ be a set of generators for M over R. Then

$$\mathrm{Ann}(M) = \bigcap_{i=1}^{r} \mathrm{Ann}(x_i),$$

meaning that there is an embedding of $R/\mathrm{Ann}(M)$ into $\prod_i R/\mathrm{Ann}(x_i)$. There is also an obvious inclusion of $R/\mathrm{Ann}(x_i)$ into M for each i (send $[r] \mapsto rx_i$), and thus we have

$$\prod_i R/\mathrm{Ann}(x_i) \hookrightarrow \prod_i M.$$

Therefore, $R/\mathrm{Ann}(M)$ injects into a product of r copies of M, so by Lemma 3.6 we have $\mathrm{d}(R/\mathrm{Ann}(M)) \leq \mathrm{d}(M^r)$. Finally, we know that $\mathrm{d}(M^r) = \mathrm{d}(M)$ by Remark 3.7, meaning that $\mathrm{d}(R/\mathrm{Ann}(M)) \leq \mathrm{d}(M)$. $\square$

We now prove Proposition 3.5.

Proof (of Proposition 3.5): To finish this proof we apply Proposition 3.8 to $R/\mathrm{Ann}(M)$ and appeal to Proposition 3.9 and the definition of $\dim(M)$. The following equalities should be clear:

$$\begin{aligned} \mathrm{d}(M) &= \mathrm{d}(R/\mathrm{Ann}(M)) \\ &= \dim(R/\mathrm{Ann}(M)) \\ &= \dim(M). \end{aligned}$$ □

We have thus established two ways of calculating $\dim(M)$ when M is a finitely generated module over a certain type of ring. The following section describes the subcategory of $\mathcal{U}$ in which we must work to apply Proposition 3.5.

3.2 The category $K_{fg} - \mathcal{U}$

Let K be a noetherian unstable algebra and let $K - \mathcal{U}$ denote the full subcategory of $\mathcal{U}$ whose objects are also modules over K. For this category, we also require the K-module structure map $K \otimes M \to M$ to be $\mathcal{A}$-linear. Further, let $K_{fg} - \mathcal{U}$ denote the full subcategory of $K - \mathcal{U}$ consisting of modules which are finitely generated over K. (These categories were used by H. W. Henn in [18].) Our immediate goal is to show that for objects of $K_{fg} - \mathcal{U}$, the nilpotent filtration is finite.

Notation When we write $K_{fg} - \mathcal{U}$ in what follows, we assume that K is a noetherian unstable algebra.

We begin our examination with the following lemma.

Lemma 3.10 *Suppose that M, M' and M'' are unstable modules which fit into a short exact sequence:*

$$0 \to M' \to M \to M'' \to 0.$$

If M' and M'' have a finite nilpotent filtration, then so does M.

PROOF: For an unstable module N, the integer $\mathrm{d}_0(N)$ was defined in [20, Section I.3.5] by the following formula:

$$\mathrm{d}_0(N) := \inf\{\ k \in \mathbb{N} \mid \lambda_{k+1,N} : N \to L_{k+1}(N) \text{ is monic}\}.$$

We have assumed that both M' and M'' have a finite nilpotent filtration, so the numbers $\mathrm{d}_0(M')$ and $\mathrm{d}_0(M'')$ are finite. Let $r \geq \max\{\mathrm{d}_0(M')+1, \mathrm{d}_0(M'')+1\}$. Since $\mathrm{nil}_r : \mathcal{U} \to \mathcal{N}il_r$ is defined as a right adjoint (see Section 2.3), it is left exact, meaning that $0 \to \mathrm{nil}_r M' \to \mathrm{nil}_r M \to \mathrm{nil}_r M''$ is exact. The number r was chosen so that $\mathrm{nil}_r M' = 0 = \mathrm{nil}_r M''$, and thus $\mathrm{nil}_r M = 0$. $\square$

We now use this lemma to prove the following proposition.

Proposition 3.11 *If an unstable module M admits a finite filtration where each composition factor has a finite nilpotent filtration, then M has a finite nilpotent filtration.*

PROOF: Let $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ be a finite filtration of M and define R_i by $R_i := M_i/M_{i-1}$. We first notice that $M_1 = R_1$ has a finite nilpotent filtration. Then, since we have a short exact sequence $0 \to M_1 \to M_2 \to R_2 \to 0$, we conclude from Lemma 3.10 that M_2 has a finite nilpotent filtration. Inductively, the short exact sequence $0 \to M_{i-1} \to M_i \to R_i \to 0$ ensures that M_i has a finite nilpotent filtration for each i. Since the filtration on M is finite, we conclude that M has a finite nilpotent filtration. □

In [18], Henn shows that each module in $K_{fg}-\mathcal{U}$ has a finite filtration where each composition factor is a multiple suspension of a reduced module. Reduced modules have a finite nilpotent filtration, so Proposition 3.11 implies that we have proved the following theorem.

Theorem 3.12 *If M is a module in $K_{fg} - \mathcal{U}$, then the nilpotent filtration of M is finite.*

Theorem 3.12 provides the first reason to focus our attention on the category K_{fg}–$\mathcal{U}$. We now show that objects of K_{fg}–$\mathcal{U}$ satisfy the conditions of Proposition 3.5, meaning that the growth and dimension of such modules are identical.

Consider an object M in $K_{fg} - \mathcal{U}$ and let K' be the largest unstable subalgebra of K which is concentrated in even dimensions. (We note that K' is sometimes denoted $\mathcal{O}\widetilde{\mathcal{O}}K$ in the literature.)

Proposition 3.13 (Lemma 5.2 in [8]) *If K is a noetherian ring, it is finitely generated as a K'-module, and K' is a noetherian ring.*

Since M is finitely generated over K, Proposition 3.13 tells us that M is finitely generated over K'. Furthermore, since K' is concentrated in even degrees, it is commutative. This means that M satisfies the conditions of Proposition 3.5, which we now restate in the present context.

Proposition 3.14 *If M is a module in $K_{fg} - \mathcal{U}$, then* $\dim(M) = \mathrm{d}(M)$.

If the nilpotent filtration of a module M is finite, there are a finite number of reduced modules $R_s(M)$ corresponding to M as defined by the formula in (2.4). In the next section we show that the modules $\overline{R}_s(M)$ inherit from M the structure necessary to discuss their dimension.

3.3 The modules $\overline{R}_s(M)$ and dimension

Let M be a module in $K_{fg} - \mathcal{U}$. Recall the notation $R_0(K) = K/\mathrm{nil}_1 K$ from Section 2.3. We want to show that $\overline{R}_0(K)$ is again noetherian and that, for every s, $\overline{R}_s(M)$ is an object in $\overline{R}_0(K)_{fg} - \mathcal{U}$. By Proposition 3.14 this will establish a good notion of the dimension of $\overline{R}_s(M)$.

We accomplish this in two steps, first showing that $R_s(M) \in R_0(K)_{fg} - \mathcal{U}$ for every s. It is clear that $R_0(K)$ is noetherian, since it is the quotient of a noetherian algebra. In order to prove that $R_s(M)$ is a finitely generated $R_0(K)$-module, we must first show that $\mathrm{nil}_s M \in K_{fg} - \mathcal{U}$. Since M is a K-module, we have a multiplication map $\mu : K \otimes M \to M$; we also recall the following formula from [21, Section 2.5]:

$$\mathrm{nil}_k\,(L \otimes N) = \sum_{i+j=k} \left(\mathrm{nil}_i\, L \otimes \mathrm{nil}_j\, N\right).$$

The following composition is the map defining the K-module structure on $\mathrm{nil}_s\, M$:

$$K \otimes \mathrm{nil}_s\, M = \mathrm{nil}_0\, K \otimes \mathrm{nil}_s\, M \xhookrightarrow{\;i\;} \mathrm{nil}_s(K \otimes M) \xrightarrow{\mathrm{nil}_s\, \mu} \mathrm{nil}_s\, M.$$

Since K is noetherian and M is finitely generated as a K-module, $\mathrm{nil}_s\, M \subseteq M$ is also finitely generated as a K-module. Thus, $\mathrm{nil}_s\, M$ is an object of $K_{fg} - \mathcal{U}$.

The next step is to show that $\Sigma^s R_s(M) \in R_0(K) - \mathcal{U}$. Applying the functor nil_{s+1} to the multiplication map μ produces

$$\mathrm{nil}_1\, K \otimes \mathrm{nil}_s\, M \xhookrightarrow{\;i\;} \mathrm{nil}_{s+1}(K \otimes M) \xrightarrow{\mathrm{nil}_{s+1}\, \mu} \mathrm{nil}_{s+1}\, M \subseteq \mathrm{nil}_s\, M.$$

This composition is the map defining the $\mathrm{nil}_1\, K$-module structure on $\mathrm{nil}_s\, M$. This shows that the action of $\mathrm{nil}_1\, K$ on $\mathrm{nil}_s\, M / \mathrm{nil}_{s+1}\, M = \Sigma^s R_s(M)$ is trivial, so $\Sigma^s R_s(M)$ is a module over $R_0(K)$. It is not difficult to see that $\Sigma^s R_s(M)$ is also finitely generated over $R_0(K)$ since $\mathrm{nil}_s\, M$ is finitely generated over K. We want to conclude that $R_s(M) \in R_0(K)_{fg} - \mathcal{U}$. We have shown that $\Sigma^s R_s(M) \in R_0(K)_{fg} - \mathcal{U}$, so this follows since the suspension of a module simply shifts the module degree-wise.

In what remains, we must show that $\overline{R}_s(M) \in \overline{R}_0(K)_{fg} - \mathcal{U}$, and to do so we draw on the work of others. The following result appears in [20, p.46] as a corollary of an important localization theorem.

Proposition 3.15 (Corollary 4.10 in [20]) *Let K be an unstable algebra.*

(i) If K is noetherian, then $L_n K$ is noetherian.

(ii) If K is noetherian and $M \in K_{fg} - \mathcal{U}$, then $L_n M \in (L_n K)_{fg} - \mathcal{U}$.

As described in Section 2.3, $L_n : \mathcal{U} \to \mathcal{U}$ is the localization functor corresponding to the subcategory $\mathcal{N}il_n$. Our result is therefore achieved by applying Proposition 3.15(ii) to $R_s(M) \in R_0(K)_{fg} - \mathcal{U}$ when $n = 1$. We summarize our work in this section in a single proposition.

Proposition 3.16 *If M is in $K_{fg} - \mathcal{U}$, then $\overline{R}_s(M)$ is an object in $\overline{R}_0(K)_{fg} - \mathcal{U}$ for every s.*

This proposition allows us to speak of the dimension of $\overline{R}_s(M)$ when M is an object in $K_{fg} - \mathcal{U}$. In the next section we will explain how to use the numbers $\dim(\overline{R}_s(M))$ to calculate $\dim(M)$.

3.4 A strategy for calculating $\dim(M)$

In what follows, we will refer to both the dimension (Definition 3.1) and growth (Definition 3.3) of a module by the word *dimension*, and we will use the notation

$\dim(M)$. For a module M in $K_{fg} - \mathcal{U}$, these terms are interchangeable by Proposition 3.5. In this section we will show that we can calculate $\dim(M)$ by examining the maximum of the set $\{\dim(\overline{R}_s(M))\}$. We begin with a proposition about the dimension of reduced modules.

Proposition 3.17 *If M is a reduced module in $K_{fg} - \mathcal{U}$, then* $\dim(M) = \dim(\overline{M})$.

Proof: Recall that the dimension of M is defined by the following equation:

$$\dim(M) := \dim(K/\text{Ann}(M)).$$

Since M is a K-module, $\overline{M}$ is also a K-module by means of $\lambda_M : M \to \overline{M}$. The proposition will follow if we can show that $\text{Ann}(M) = \text{Ann}(\overline{M})$ when M is reduced.

We first prove that $\text{Ann}(M)$ is a sub-$\mathcal{A}$-module of K. If we begin with $\theta \in \mathcal{A}$ and $k \in \text{Ann}(M)$, we need to show that $\theta k \in \text{Ann}(M)$. We proceed by induction on the dimension of θ. When $|\theta| = 0$, there is nothing to prove because θ is the identity. The key step in the remaining argument is the following fact, which can be verified using the Cartan formula.

Fact 3.18 If $\theta \in \mathcal{A}$, $k \in K$ and $m \in M$, then

$$\theta(km) = (\theta k)m \; + \sum_{\substack{|\theta_i|+|\theta_j|=|\theta| \\ |\theta_i|<|\theta|}} (\theta_i k)(\theta_j m). \tag{3.1}$$

Since $km = 0$, we know that $\theta(km) = 0$. The inductive hypothesis says that

$\theta' k \in \mathrm{Ann}(M)$ whenever $|\theta'| < |\theta|$. From (3.1) we have $|\theta_i| < |\theta|$, so $(\theta_i k)(\theta_j m) = 0$. Therefore, $(\theta k)m = 0$, meaning $\theta k \in \mathrm{Ann}(M)$.

Consider the map $\mu : K \otimes M \to M$ which defines the K-module structure on M. If $k \in \mathrm{Ann}(M)$, the composition

$$\mathcal{A}k \otimes M \overset{i}{\hookrightarrow} K \otimes M \xrightarrow{\mu} M$$

is zero. Further, since M is reduced, $\lambda_M : M \to \overline{M}$ is monic and $\mu \circ i = 0$ if and only if $\lambda_M \circ \mu \circ i = 0$. This shows the following:

$$\begin{aligned} \mathrm{Ann}(M) &= \{k \in K \mid \mu \circ i : \mathcal{A}k \otimes M \to M \text{ is zero}\} \\ &= \{k \in K \mid \lambda_M \circ \mu \circ i : \mathcal{A}k \otimes M \to \overline{M} \text{ is zero}\}. \end{aligned} \tag{3.2}$$

If we denote the maps which define the K-module structure of M and $\overline{M}$ by μ and $\overline{\mu}$, respectively, where $\overline{\mu} = \lambda_M \circ \mu$, the following diagram commutes for all $k \in K$:

$$\begin{array}{ccccc} \mathcal{A}k \otimes M & \overset{i}{\hookrightarrow} & K \otimes M & \xrightarrow{\mu} & M \\ \downarrow{\scriptstyle \mathrm{id} \otimes \lambda_M} & & \downarrow{\scriptstyle \mathrm{id} \otimes \lambda_M} & & \downarrow{\scriptstyle \lambda_M} \\ \mathcal{A}k \otimes \overline{M} & \underset{i'}{\hookrightarrow} & K \otimes \overline{M} & \xrightarrow[\overline{\mu}]{} & \overline{M}. \end{array} \tag{3.3}$$

We would like to show that for a given $k \in K$, $\lambda_M \circ \mu \circ i = 0$ if and only if $\overline{\mu} \circ i' = 0$. One of these directions is easy to see, for if $\overline{\mu} \circ i' = 0$, then $\overline{\mu} \circ i' \circ (\mathrm{id} \otimes \lambda_M) = 0$, and thus $\lambda_M \circ \mu \circ i = 0$ by the commutativity of (3.3). The other implication will be straightforward after the following fact, which is true by the universal property of $\mathcal{N}il$-closure.

Fact 3.19 Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be morphisms in $\mathcal{U}$. If f is a $\mathcal{N}il$-epimorphism and M_3 is $\mathcal{N}il$-closed, then $g \circ f = 0$ implies $g = 0$.

We recognize that since $\mathrm{id} \otimes \lambda_M$ is a $\mathcal{N}il$-isomorphism and $\overline{M}$ is $\mathcal{N}il$-closed, Fact 3.19 shows that if $\overline{\mu} \circ i' \circ (\mathrm{id} \otimes \lambda_M) = 0$ then $\overline{\mu} \circ i' = 0$. However, by the commutativity of (3.3), this is exactly what we get if we assume $\lambda_M \circ \mu \circ i = 0$. We have therefore proven the following:

$$\begin{aligned} \mathrm{Ann}(\overline{M}) &= \{k \in K \mid \overline{\mu} \circ i' : \mathcal{A}k \otimes \overline{M} \to \overline{M} \text{ is zero}\} \\ &= \{k \in K \mid \lambda_M \circ \mu \circ i : \mathcal{A}k \otimes M \to \overline{M} \text{ is zero}\}. \end{aligned} \tag{3.4}$$

Finally, a comparison of (3.2) and (3.4) reveals that $\mathrm{Ann}(M) = \mathrm{Ann}(\overline{M})$, meaning that $\dim(M) = \dim(\overline{M})$, as desired. □

For contextual reasons, we present the following proposition now but reserve its use until Section 4.2.

Proposition 3.20 *Let A and B be modules in $K_{fg} - \mathcal{U}$, and let B be a reduced module. If $f : A \to B$ is an $\mathcal{N}il$-epimorphism, then* $\dim(A) \geq \dim(B)$.

Proof: Since B is reduced, $f(A)$ is also reduced, meaning $\dim(B) = \dim(\overline{B})$ and $\dim(f(A)) = \dim(\overline{f(A)})$ by Proposition 3.17. Since f is a $\mathcal{N}il$-epimorphism, we see that $f(A) \hookrightarrow B$ is a $\mathcal{N}il$-isomorphism, giving $\overline{f(A)} \cong \overline{B}$. This means that $\dim(\overline{f(A)}) = \dim(\overline{B})$, and thus $\dim(f(A)) = \dim(B)$. Since Lemma 3.6(i) implies $\dim(A) \geq \dim(f(A))$, the proposition follows. □

We now have the proper tools to state and prove the key theorem of this section. This summarizes the method of calculating dimension which we employ in the rest of this work.

Theorem 3.21 *If M is a module in $K_{fg} - \mathcal{U}$, then* $\dim(M) = \max\{\dim(\overline{R}_s(M))\}$.

PROOF: For this argument, let $\mathrm{d}' := \max\{\dim(\overline{R}_s(M))\}$. Additionally, define a third number $\mathrm{d}''(M)$ by $\mathrm{d}''(M) := \max\{\dim(R_s(M))\}$. We have remarked earlier (see Section 3.3) that $\overline{R}_0(K)$ is a noetherian unstable algebra, and since $R_s(M)$ is a reduced module in $\overline{R}_0(K)_{fg} - \mathcal{U}$ for every s, Proposition 3.17 implies that $\mathrm{d}'(M) = \mathrm{d}''(M)$. We will prove Theorem 3.21 by showing $\dim(M) = \mathrm{d}''(M)$.

To prove that $\mathrm{d}''(M) \leq \dim(M)$, we first take note of the short exact sequence $0 \to \operatorname{nil}_{s+1} M \to \operatorname{nil}_s M \to \Sigma^s R_s(M) \to 0$ for each s. Lemma 3.6(i) implies that $\dim(\operatorname{nil}_s M) \leq \dim(M)$ for every s. Then, since $\dim(R_s(M)) = \dim(\Sigma^s R_s(M))$ and $\dim(\Sigma^s R_s(M)) \leq \dim(\operatorname{nil}_s M)$ for each s, we have $\dim(R_s(M)) \leq \dim(M)$ for all s, and the inequality we desire is immediate.

Showing $\dim(M) \leq \mathrm{d}''(M)$ requires repeated application of Lemma 3.6(i). We argue by contradiction and suppose that $\dim(M) > \dim(R_s(M))$ for all s. The short exact sequence

$$0 \to \operatorname{nil}_1 M \to M \to R_0(M) \to 0$$

then implies that $\dim(\operatorname{nil}_1 M) = \dim(M)$, and subsequent short exact sequences

$$0 \to \operatorname{nil}_{i+1} M \to \operatorname{nil}_i M \to \Sigma^i R_i(M) \to 0$$

show that $\dim(\operatorname{nil}_s M) = \dim(M)$ for all s. Since the nilpotent filtration of M is finite by Theorem 3.12, there exists a number t such that $\operatorname{nil}_t M \neq 0$ but $\operatorname{nil}_{t+1} M = 0$. This means that $\operatorname{nil}_t M = \Sigma^t R_t(M)$, and this contradicts our earlier assumption that $\dim(R_s(M)) < \dim(M)$ for all s. This shows that $\dim(M) \leq \mathrm{d}''(M)$, meaning $\dim(M) = \mathrm{d}''(M)$, as desired. □

We will use Theorem 3.21 in the next chapter as our primary method of calculating dimension. We end this chapter by presenting an important example.

3.5 An example

In this section we include an example which shows that the approach we have outlined in Section 3.4 can be used to answer Question 1.9.

Let P be a finite p-group and let S be a simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module. Let e_S be the idempotent in $\mathbb{F}_p[\mathrm{Out}(P)]$ which corresponds to S. We will describe the noetherian unstable algebra K which provides the category $K_{fg} - \mathcal{U}$ for the module $e_S H^*(P)$.

The classical Dickson invariants, $\mathbb{F}_p[x_1, \ldots, x_n]^{\mathrm{GL}_n(\mathbb{F}_p)}$, were calculated by Dickson to be a polynomial algebra on generators $u_1, \ldots, u_n$, where u_i has dimension $2(p^n - p^{n-i})$. If V is an elementary abelian p-group of rank n, then the polynomial part of $H^*(V)$, denoted $P(V)$, has dimension n, and we can describe the fixed points $P(V)^{\mathrm{GL}(V)}$ as above, where the generators for the polynomial algebra are the images of the Chern classes $\{c_{p^n-p^i} \mid i = 0, 1, \ldots, n-1\}$ under the regular representation $\mathrm{reg}_V : V \to U(p^n)$ ([35]). For such an elementary abelian p-group, we will describe $P(V)^{\mathrm{GL}(V)}$ by $\mathbb{F}_p[v_{p^n-p^i} \mid i = 0, 1, \ldots, n-1]$. In [26, Section 1], Martino and Priddy generalize this discussion to all p-groups.

Definition 3.22 Let P be a p-group of order p^r and let $\mathrm{r}_p(P) = n$. Consider the regular representation

$$\mathrm{reg}_P : P \to U(p^r).$$

Then the *Dickson invariants* are defined to be

$$D(P) := \mathrm{reg}_P^* \, C(n) \subseteq H^*(P),$$

where $C(n) = \mathbb{F}_p[c_{p^s(p^n-p^i)} \mid i = 0, 1, \ldots, n-1\} \subseteq H^*(BU(p^r))$ and $s = r - n$.

Martino and Priddy prove several facts about $D(P)$.

1. $D(P)$ is a polynomial algebra of dimension n.

2. If V is an elementary abelian p-subgroup of P of maximal rank, then under $\mathrm{res}^P_V : H^*(P) \to H^*(V)$, $D(P)$ has image $\mathbb{F}_p[(v_{p^n - p^i})^{p^s} \mid i = 0, 1, \ldots, n-1]$.

3. The elements of $D(P)$ are $\mathrm{Out}(P)$-invariant.

4. $H^*(P)$ is a finitely generated $D(P)$-module.

We can finish this example using these properties of $D(P)$. Since $D(P)$ is a noetherian ring and $H^*(P)$ is finitely generated over $D(P)$, $H^*(P)$ is a noetherian $D(P)$-module. Then, since $H^*(P)^{\mathrm{Out}(P)}$ is a $D(P)$-submodule of $H^*(P)$, it is also finitely generated over $D(P)$, meaning that $H^*(P)^{\mathrm{Out}(P)}$ is a noetherian algebra. This shows that $e_S H^*(P)$ is an object in $H^*(P)^{\mathrm{Out}(P)}_{fg} - \mathcal{U}$, and we can therefore determine $\dim(e_S H^*(P))$ by calculating $\dim(\overline{R}_n(e_S H^*(P)))$ for each n.

We will discuss the situation described in this example in the following chapter. The framework we have established here will enable us to calculate $\dim(e_S(H^*(P)))$ and thus provide an answer to Question 1.9.

Chapter 4

The dimension of $eH^*(P)$

Our goal in this chapter is to present an initial answer to Question 1.9. In Section 4.1 we relate a notion of maximality on the subgroups of P to the question of dimension. Section 4.2 describes the way in which the presence of the simple module S as a composition factor in certain modules ensures a lower bound for $\dim(e_S H^*(P))$. Finally, we set forth the group theoretic conditions under which we have an answer to Question 1.9 in Section 4.3.

4.1 Maximality and composition factors

In this section we will discuss some of the structure of p-groups. In particular, we want to develop the proper notion of "maximal" subgroups, so that we can make conclusions about dimension in subsequent sections.

We first define an equivalence relation on the subgroups of P. For subgroups V_1 and V_2 of P, we say that $V_1 \sim V_2$ if there exists $\alpha \in \mathrm{Aut}(P)$ with $\alpha(V_1) = V_2$. We will denote the equivalence class of V by $[V]$. It is easy to define an ordering on these equivalence classes: we write $[V] \leq [V']$ if there exists a subgroup $V'' \leq V'$ with $V \sim V''$.

For the next lemma, we need to recall the action of $\mathrm{Out}(P)$ on $\mathrm{Rep}(W, P)$, where W is an elementary abelian p-group. For $[f] \in \mathrm{Out}(P)$ and $[\gamma] \in \mathrm{Rep}(W, P)$, the action is $[f] \cdot [\gamma] := [f \circ \gamma]$. The $\mathrm{Out}(P)$-stabilizer of $[\gamma]$ will be written $\mathrm{Out}(P)_{[\gamma]}$.

Lemma 4.1 *Let A and B be finite groups, and let α and β be elements of* $\mathrm{Hom}(A, B)$ *with $\alpha(A) = \beta(A)$. Then* $\mathrm{Out}(B)_{[\alpha]} = \mathrm{Out}(B)_{[\beta]}$, *where $[\alpha]$ and $[\beta]$ denote the classes of α and β (respectively) in* $\mathrm{Rep}(A, B)$.

Proof: Let $[f] \in \mathrm{Out}(B)_{[\alpha]}$, so that $[f] \cdot [\alpha] = [\alpha]$. This means that there exists an element $b \in B$ such that $f \circ \alpha = c_b \circ \alpha$. Let $a \in A$. Then, since $\alpha(A) = \beta(A)$, there exists $a' \in A$ such that $\alpha(a') = \beta(a)$, meaning

$$\begin{aligned} f(\beta(a)) &= f(\alpha(a')) \\ &= c_b(\alpha(a')) \\ &= c_b(\beta(a)). \end{aligned}$$

This shows that $[f] \cdot [\beta] = [\beta]$, so $[f] \in \mathrm{Out}(B)_{[\beta]}$. The argument for the opposite inclusion is identical, and we conclude that $\mathrm{Out}(B)_{[\alpha]} = \mathrm{Out}(B)_{[\beta]}$. □

Lemma 4.1 eliminates ambiguity from the following notation.

Notation Let α_V be any map in $\mathrm{Hom}(W, P)$ with image V. Then we define $\mathrm{Out}(P)_V$ by

$$\mathrm{Out}(P)_V := \mathrm{Out}(P)_{[\alpha_V]}.$$

This definition is also independent of the choice of W. We will use this notation later when referring to an $\mathrm{Out}(P)$-orbit of $\mathrm{Rep}(W, P)$ as $\mathrm{Out}(P)\cdot[\alpha_V] = \mathrm{Out}(P)/\mathrm{Out}(P)_V$.

Recall that if S is a simple R-module and e_S is the corresponding idempotent in R, then $e_S M = 0$ if and only if S does not appear as a composition factor in M. In the following proposition we show that for certain $\mathbb{F}_p[\mathrm{Out}(P)]$-modules, there is a relationship between the presence of composition factors and the ordering on the equivalence classes of subgroups.

Proposition 4.2 *Let U and V be subgroups of P. If $e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$ and $[V] \leq [U]$, then $e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_U} \neq 0$.*

Proof: Since $[V] \leq [U]$, there exists $\alpha \in \mathrm{Aut}(P)$ and a subgroup $V_1 \leq U$ with $\alpha(V) = V_1$. Since $V_1 \leq U$, we have $\mathrm{Out}(P)_U \subseteq \mathrm{Out}(P)_{V_1}$, providing a surjection

$$\mathrm{Out}(P)/\mathrm{Out}(P)_U \twoheadrightarrow \mathrm{Out}(P)/\mathrm{Out}(P)_{V_1}. \tag{4.1}$$

Let $\phi \in \mathrm{Hom}(W, P)$ with $\phi(W) = V$, and let $\beta = \alpha \circ \phi$, so that $\beta(W) = V_1$. Since $[\beta]$ and $[\phi]$ are in the same $\mathrm{Out}(P)$-orbit of $\mathrm{Rep}(W, P)$, there is a bijective correspondence between the sets $\mathrm{Out}(P)/\mathrm{Out}(P)_{[\beta]}$ and $\mathrm{Out}(P)/\mathrm{Out}(P)_{[\phi]}$. Then, since $\mathrm{Out}(P)_{[\beta]} = \mathrm{Out}(P)_{V_1}$ and $\mathrm{Out}(P)_{[\phi]} = \mathrm{Out}(P)_V$, we see that there is also a bijective correspondence between the sets $\mathrm{Out}(P)/\mathrm{Out}(P)_V$ and $\mathrm{Out}(P)/\mathrm{Out}(P)_{V_1}$, so that (4.1) can be realized as

$$\mathrm{Out}(P)/\mathrm{Out}(P)_U \twoheadrightarrow \mathrm{Out}(P)/\mathrm{Out}(P)_V.$$

This extends linearly to a surjection

$$\mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_U] \twoheadrightarrow \mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_V],$$

meaning that every composition factor in $\mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_V]$ also appears as a composition factor in $\mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_U]$.

Now, $\mathbb{F}_p[X]$ and $\mathbb{F}_p^X$ are dual modules for any finite set X. So, if S is a composition factor for $\mathbb{F}_p[X]$, its dual S^* is a composition factor for $\mathbb{F}_p^X$. (This argument also works the other way.) Therefore, the argument in the previous paragraph implies that every composition factor of $\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V}$ is a composition factor of $\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_U}$, proving the proposition. □

This proposition tells us that if $e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$, we can extend that knowledge to any class of subgroups $[U]$ with $[V] \leq [U]$. We will relate this result to dimension in the next section.

4.2 A lower bound for $\dim(e_S H^*(P))$

Our current goal is to describe conditions under which we have a lower bound for the dimension of $e_S H^*(P)$. We will first explore relationships between functors in $\mathcal{F}$ and then translate the situation back to $\mathcal{U}$.

In this section, W will denote an elementary abelian p-group. We will continue to use the equivalence relation on subgroups of P described at the beginning of Section 4.1. These equivalence classes of subgroups of P provide a tool for a stratification of $\mathrm{Rep}(W, P)$. For an elementary abelian p-subgroup V of P, define $\mathrm{Rep}(W, P)_V$ by

$$\mathrm{Rep}(W, P)_V := \{[\phi] \in \mathrm{Rep}(W, P) \mid \phi(W) \sim V\}.$$

We see that $\mathrm{Out}(P)$ acts on $\mathrm{Rep}(W, P)_V$ in the usual way, and the following decomposition of $\mathrm{Out}(P)$-sets is clear:

$$\mathrm{Rep}(W, P) = \coprod_{[V]} \mathrm{Rep}(W, P)_V.$$

As a consequence, we have

$$\mathbb{F}_p^{\mathrm{Rep}(W,P)} = \bigoplus_{[V]} \mathbb{F}_p^{\mathrm{Rep}(W,P)_V}, \tag{4.2}$$

as $\mathrm{Out}(P)$-modules.

Lemma 4.3 *In any* $\mathrm{Out}(P)$*-orbit of* $\mathrm{Rep}(W,P)_V$*, there is an element* $[\phi]$ *containing a homomorphism* ϕ *such that* $\phi(W) = V$*.*

Proof: Let $\mathrm{Out}(P) \cdot [\phi]$ be an $\mathrm{Out}(P)$-orbit of $\mathrm{Rep}(W,P)_V$. Since $\phi(W) \sim V$, there is an automorphism $\alpha \in \mathrm{Aut}(P)$ such that $\alpha(\phi(W)) = V$. Now, $[\alpha \circ \phi]$ is the element of $\mathrm{Rep}(W,P)_V$ in the same $\mathrm{Out}(P)$-orbit as $[\phi]$ with the desired property. □

Let $\mathrm{Epi}(W,V)$ denote the set of group epimorphisms from W to V. Define $\mathrm{Aut}(P)_V$ by

$$\mathrm{Aut}(P)_V := \{\alpha \in \mathrm{Aut}(P) \mid \alpha(V) = V\},$$

and notice that there is a natural action of $\mathrm{Aut}(P)_V$ on $\mathrm{Epi}(W,V)$. There is an obvious map $\mathrm{Aut}(P)_V \to \mathrm{GL}(V)$ which sends an automorphism to its restriction to V; define $\mathrm{GL}(V)_P$ to be the image of this map. We can now define an action of $\mathrm{GL}(V)_P$ on $\mathrm{Epi}(W,V)$: if $g \in \mathrm{GL}(V)_P$, $\alpha \in \mathrm{Epi}(W,V)$, and $\overline{g}$ is any element of $\mathrm{Aut}(P)_V$ in the preimage of g, then $g \cdot \alpha := \overline{g} \circ \alpha$. This action is clearly well-defined, since any two maps in the preimage of g will agree when restricted to V. Denote the orbit of $\alpha \in \mathrm{Epi}(W,V)$ under this action by $[\alpha]_G$.

Since $\mathrm{Rep}(W,P)_V$ is an $\mathrm{Out}(P)$-set, it decomposes as the disjoint union of its orbits under this action. In the following lemma, we count the number of these orbits.

Lemma 4.4 *The following is an equality of* $\mathrm{Out}(P)$*-sets:*

$$\mathrm{Rep}(W,P)_V = \coprod_{\mathrm{Epi}(W,V)/\,\mathrm{GL}(V)_P} \mathrm{Out}(P)/\mathrm{Out}(P)_V.$$

Proof: We wish to prove the existence of a bijective correspondence between the set X of $\mathrm{Out}(P)$-orbits of $\mathrm{Rep}(W,P)_V$ and the set Y of $\mathrm{GL}(V)_P$-orbits of $\mathrm{Epi}(W,V)$. We will describe maps between these sets, show that both maps are well-defined, and then verify that the composition in both directions is the identity. This will prove the lemma.

We will first discuss the map $f : X \to Y$. By Lemma 4.3, for each orbit $\mathrm{Out}(P)\cdot[\phi]$ there exists an element of $\mathrm{Rep}(W,P)_V$ which has a representative with image equal to V. For the orbit $\mathrm{Out}(P)\cdot[\phi]$, call this element $[\phi_V]$. Then ϕ_V can be considered as an element of $\mathrm{Epi}(W,V)$, so we will declare that $f(\mathrm{Out}(P)\cdot[\phi]) := [\phi_V]_G$. The map in the other direction, $g : Y \to X$, is much easier to explain; $g([\phi]_G) := \mathrm{Out}(P)\cdot[\phi]$.

There are several choices in the definition of these two maps. Suppose that in the orbit $\mathrm{Out}(P)\cdot[\phi]$ there were two elements $[\phi_V]$ and $[\psi_V]$ which have representatives with image V; we must show that $[\phi_V]_G = [\psi_V]_G$. Since $[\phi_V]$ and $[\psi_V]$ are in the same $\mathrm{Out}(P)$-orbit, there is an element $[f] \in \mathrm{Out}(P)$ such that $[f]\cdot[\phi_V] = [\psi_V]$. This means that there is an element $p \in P$ such that $c_p \circ f \circ \phi_V = \psi_V$, which proves that $c_p \circ f$ is a map in $\mathrm{Aut}(P)_V$ giving $[\phi_V]_G = [\psi_V]_G$.

It is also easy to establish that the map g is well-defined. If $[\phi_1]_G = [\phi_2]_G$, we need to show that $[\phi_1]$ and $[\phi_2]$ are in the same $\mathrm{Out}(P)$-orbit of $\mathrm{Rep}(W,P)_V$. Since $[\phi_1]_G = [\phi_2]_G$, there is an element $m \in \mathrm{GL}(V)_P$ such that $m \cdot \phi_1 = \phi_2$. If $\overline{m}$ is an element of $\mathrm{Aut}(P)_V$ in the preimage of m, then $\overline{m} \circ \phi_1 = \phi_2$. It is now easy to see that $[\phi_1] = [\phi_2]$, since $[\overline{m}] \cdot [\phi_1] = [\phi_2]$.

It is immediate that $g \circ f$ is the identity. In proving that the composition in the other order is the identity, we need to check that $[\phi]_G = [\phi_V]_G$. Since $[\phi]$ and $[\phi_V]$ are in the same $\mathrm{Out}(P)$-orbit, there exists an $[f]$ in $\mathrm{Out}(P)$ such that $[f] \cdot [\phi] = [\phi_V]$. This means that $c_p \circ f \circ \phi = \phi_V$ for some $p \in P$, and we see that $[\phi]_G = [\phi_V]_G$ since $c_p \circ f \in \mathrm{Aut}(P)_V$. □

Another way to state the conclusion of Lemma 4.4 is to say that the following is an equality of $\mathrm{Out}(P)$-sets:

$$\mathrm{Rep}(W,P)_V = \mathrm{Epi}(W,V)/\,\mathrm{GL}(V)_P \times \mathrm{Out}(P)/\mathrm{Out}(P)_V.$$

As an immediate consequence, we have the following equality of $\mathrm{Out}(P)$-modules:

$$\mathbb{F}_p^{\mathrm{Rep}(W,P)_V} = \mathbb{F}_p^{\mathrm{Epi}(W,V)/\,\mathrm{GL}(V)_P} \otimes \mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V}. \tag{4.3}$$

We would like to conclude that the equality in (4.3) is true both as $\mathrm{Out}(P)$-modules and as functors of W, which requires a description of the functors involved. The action of $\mathbb{F}_p^{\mathrm{Rep}(-,P)_V}$ on objects is clear, so we describe its action on morphisms. Given a morphism in $\mathcal{E}$, $\alpha : W \to W'$, we must detail the map

$\alpha_* : \mathbb{F}_p^{\mathrm{Rep}(W,P)_V} \to \mathbb{F}_p^{\mathrm{Rep}(W',P)_V}$. If f is a map in $\mathbb{F}_p^{\mathrm{Rep}(W,P)_V}$ and $[\phi]$ is an element of $\mathrm{Rep}(W',P)_V$, then

$$\alpha_*(f)([\phi]) = \begin{cases} f([\phi\circ\alpha]) & \text{if } (\phi\circ\alpha)(W') \sim V \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to check that the functor axioms are satisfied by this definition.

We now describe the way in which $\mathbb{F}_p^{\mathrm{Epi}(-,V)/\,\mathrm{GL}(V)_P}$ is a functor. Given a morphism in $\mathcal{E}$, $\alpha : W \to W'$, we need to describe $\alpha_* : \mathbb{F}_p^{\mathrm{Epi}(W,V)/\,\mathrm{GL}(V)_P} \to \mathbb{F}_p^{\mathrm{Epi}(W',V)/\,\mathrm{GL}(V)_P}$. Let g be an element of $\mathbb{F}_p^{\mathrm{Epi}(W,V)/\,\mathrm{GL}(V)_P}$ and let $[\beta]_G \in \mathrm{Epi}(W',V)/\,\mathrm{GL}(V)_P$. Then,

$$\alpha_*(g)([\beta]_G) = \begin{cases} g([\beta\circ\alpha]_G) & \text{if } \beta\circ\alpha \in \mathrm{Epi}(W,V) \\ 0 & \text{otherwise.} \end{cases}$$

This discussion verifies that (4.3) is true both as $\mathrm{Out}(P)$-modules and as functors of W. Consequently, as we combine (4.2) and (4.3) we see that we have the following equality in the category $\mathcal{F}$:

$$\mathbb{F}_p^{\mathrm{Rep}(W,P)} = \bigoplus_{[V]} \mathbb{F}_p^{\mathrm{Epi}(W,V)/\,\mathrm{GL}(V)_P} \otimes \mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V}. \tag{4.4}$$

We previously described the sets $\mathrm{Rep}(W,P)_V$ as a way to stratify $\mathrm{Rep}(W,P)$. We now introduce a filtration of $\mathrm{Rep}(W,P)$ which also depends on rank. For each integer

k, $0 \leq k \leq \mathrm{r}_p(P)$, define $F_k\mathrm{Rep}(W, P)$ by

$$F_k\mathrm{Rep}(W, P) := \{[\phi] \in \mathrm{Rep}(W, P) \mid \mathrm{rk}(\phi(W)) \leq k\}.$$

We then have the following increasing filtration on $\mathrm{Rep}(W, P)$:

$$F_0\mathrm{Rep}(W, P) \subseteq F_1\mathrm{Rep}(W, P) \subseteq \cdots \subseteq F_{\mathrm{r}_p(P)}\mathrm{Rep}(W, P) = \mathrm{Rep}(W, P).$$

This produces a sequence of epimorphisms on the corresponding $\mathrm{Out}(P)$-modules:

$$\mathbb{F}_p^{\mathrm{Rep}(W,P)} \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{F}_p^{F_1\mathrm{Rep}(W,P)} \twoheadrightarrow \mathbb{F}_p^{F_0\mathrm{Rep}(W,P)}. \tag{4.5}$$

We conclude that for every k, $0 < k \leq \mathrm{r}_p(P)$, we have a short exact sequence:

$$0 \to \bigoplus_{\substack{[V] \\ \dim(V)=k}} \mathbb{F}_p^{\mathrm{Rep}(W,P)_V} \to \mathbb{F}_p^{F_k\mathrm{Rep}(W,P)} \to \mathbb{F}_p^{F_{k-1}\mathrm{Rep}(W,P)} \to 0. \tag{4.6}$$

We will now state the major result of this section.

Theorem 4.5 *Let S be a simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module and let e_S be the corresponding idempotent. Then if $e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$, $\dim(e_S H^*(P)) \geq \mathrm{rk}(V)$.*

We will prove Theorem 4.5 through a series of propositions. We first make the observation that if e_S is an idempotent in $\mathbb{F}_p[\mathrm{Out}(P)]$, the sequence in (4.6) is transformed to the following short exact sequence:

$$0 \to \bigoplus_{\substack{[V] \\ \dim(V)=k}} e_S\mathbb{F}_p^{\mathrm{Rep}(W,P)_V} \to e_S\mathbb{F}_p^{F_k\mathrm{Rep}(W,P)} \to e_S\mathbb{F}_p^{F_{k-1}\mathrm{Rep}(W,P)} \to 0. \tag{4.7}$$

Now, the assumption of Theorem 4.5 is that $e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$. Suppose that $\mathrm{rk}(V) = k$; if we consider both (4.3) and (4.7), we see that

$$\mathbb{F}_p^{\mathrm{Epi}(W,V)/\,\mathrm{GL}(V)_P} \otimes e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \subseteq\ e_S\mathbb{F}_p^{F_k\mathrm{Rep}(W,P)}. \tag{4.8}$$

We also see from (4.5) that we have an epimorphism

$$e_S\mathbb{F}_p^{\mathrm{Rep}(W,P)} \twoheadrightarrow e_S\mathbb{F}_p^{F_k\mathrm{Rep}(W,P)}. \tag{4.9}$$

Our strategy is to apply the functor m (see Section 2.3.2) to the situations we have just described; there will be submodules of large enough dimension in what results to prove the theorem. This approach is very similar to that which Henn, Lannes, and Schwartz use in [19, Sections II.3 and II.6]. We will calculate two specific values of m in the next two propositions.

We must include one more piece of notation. For an elementary abelian p-group V, define ω_V by

$$\omega_V := \prod_{\alpha \in V^* \smallsetminus 0} \alpha,$$

if $p=2$. If $p>2$, define ω_V by

$$\omega_V := \prod_{\alpha\in V^*\smallsetminus 0} \beta(\alpha)\,.$$

Recalling the discussion of the classic Dickson invariants from Section 3.5, we note that ω_V is the top Dickson invariant for $P(V)^{\mathrm{GL}(V)}$ (or $H^*(V)^{\mathrm{GL}(V)}$ if $p=2$).

Proposition 4.6 *If G is a subgroup of* $\mathrm{GL}(V)$*, then*

$$\omega_V H^*(V)^G \subseteq\ m\left(\mathbb{F}_p^{\mathrm{Epi}(W,V)/G}\right).$$

PROOF: We first consider the map

$$\mathbb{F}_p^{\mathrm{Hom}(W,V)} \xrightarrow{\ \mathrm{R}\ } \prod_{\substack{U<V\\ \mathrm{codim}(U)=1}} \mathbb{F}_p^{\mathrm{Hom}(W,U)}\,,$$

whose component maps

$$\mathrm{R}_U : \mathbb{F}_p^{\mathrm{Hom}(W,V)} \longrightarrow \mathbb{F}_p^{\mathrm{Hom}(W,U)}$$

are the obvious restrictions. That is, for each $f\in\mathbb{F}_p^{\mathrm{Hom}(W,V)}$, $\mathrm{R}_U(f)$ is defined by the composition

$$\mathrm{R}_U(f) : \mathrm{Hom}(W,U) \hookrightarrow \mathrm{Hom}(W,V) \xrightarrow{\ f\ } \mathbb{F}_p.$$

Claim The kernel of R is $\mathbb{F}_p^{\mathrm{Epi}(W,V)}$.

PROOF OF CLAIM: Let E be the set of all $f \in \mathbb{F}_p^{\mathrm{Hom}(W,V)}$ which have the property that $f(\alpha) = 0$ if α is not an epimorphism. There is an obvious identification of E with $\mathbb{F}_p^{\mathrm{Epi}(W,V)}$. We will prove that $E = \ker(\mathrm{R})$. If $f \in E$, then $\mathrm{R}_U(f) = 0$ for every U, since the inclusion of $\mathrm{Hom}(W,U)$ into $\mathrm{Hom}(W,V)$ means that f will be acting on a homomorphism which is by necessity not an epimorphism. This shows that $f \in \ker(\mathrm{R})$.

Let f be an element of $\ker(\mathrm{R})$. Let $\alpha : W \to V$ be a homomorphism which is not onto, and denote its image by U''. Since V is an elementary abelian p-group, we know that U'' must be contained in a codimension 1 subgroup; call that U'. Now, since $\mathrm{R}(f) = 0$, we have $\mathrm{R}_U(f) = 0$ for all $U < V$ of codimension 1. In particular, $\mathrm{R}_{U'}(f) = 0$. So, since the composition

$$\begin{array}{ccccc} \mathrm{Hom}(W,U') & \hookrightarrow & \mathrm{Hom}(W,V) & \xrightarrow{f} & \mathbb{F}_p \\ \alpha & \longmapsto & \alpha & \longmapsto & f(\alpha) \end{array}$$

must be zero, we see that $f(\alpha) = 0$. $\Diamond$

The proof of this claim shows that the following sequence is exact:

$$\mathbb{F}_p^{\mathrm{Epi}(W,V)} \longrightarrow \mathbb{F}_p^{\mathrm{Hom}(W,V)} \xrightarrow{\mathrm{R}} \prod_{\substack{U<V \\ \mathrm{codim}(U)=1}} \mathbb{F}_p^{\mathrm{Hom}(W,U)}. \tag{4.10}$$

We next consider the map

$$H^*(V) \xrightarrow{\mathrm{S}} \prod_{\substack{U<V \\ \mathrm{codim}(U)=1}} H^*(U),$$

whose components $\mathrm{S}_U : H^*(V) \to H^*(U)$ are the familiar restriction maps induced by subgroup inclusion.

Claim The kernel of S contains $\omega_V H^*(V)$.

Proof of Claim: We will prove that ω_V is sent to zero by $\mathrm{res}^V_U : H^*(V) \to H^*(U)$ for every subgroup $U \leq V$ of codimension 1, and this will verify the claim. We use the following description of $H^1(V;\mathbb{Z}/p)$:

$$H^1(V;\mathbb{Z}/p) \ \cong \ V^* = \mathrm{Hom}(V, \mathbb{Z}/p).$$

If $U < V$ and $f \in H^1(V)$, the map $\mathrm{res}^V_U : H^1(V) \to H^1(U)$ takes f to $\mathrm{res}^V_U(f)$, where $\mathrm{res}^V_U(f)$ is defined by

$$\mathrm{res}^V_U(f) : U \hookrightarrow V \xrightarrow{f} \mathbb{Z}/p.$$

If U is a codimension 1 subgroup of V, U can be realized as the kernel of a nonzero element of V^*; call this α_U. Since V is an elementary abelian p-group, one can simply extend a basis of U to a basis of V, and α_U is projection onto the last coordinate. It is then easy to see that $\mathrm{res}^V_U(\alpha_U) = 0$. Further, since res^V_U is a map of modules over the Steenrod algebra, $\mathrm{res}_U^V(\beta(\alpha_U)) = \beta(\mathrm{res}_U^V(\alpha_U))$. Then, since α or $\beta(\alpha)$ (depending on the prime) is a factor of ω_V, and since res_U^V is a ring homomorphism, we see that $\mathrm{res}_U^V(\omega_V) = 0$. This argument holds for every $U < V$ of codimension 1, so we have shown that $\omega_V \in \ker(\mathrm{S})$, meaning $\omega_V H^*(V) \subseteq \ker(\mathrm{S})$. $\Diamond$

As we examine the action of the functor m on (4.10), we rely on the following fact: since m is defined to be a right adjoint, it preserves all limits (see [23, p.118]). This means that m preserves products, kernels, equalizers, and intersections, since these constructions can be expressed as limits.

It is clear from [19, p.1058] that

$$m\left(\mathbb{F}_p^{\mathrm{Hom}(W,V)}\right) = H^*(V)$$

for any elementary abelian p-group V, so $m(\mathrm{R}) = \mathrm{S}$, and since m preserves kernels, we have $m(\mathbb{F}_p^{\mathrm{Epi}(W,V)}) = \ker(\mathrm{S})$. The argument above thus implies that

$$\omega_V H^*(V) \subseteq m\left(\mathbb{F}_p^{\mathrm{Epi}(W,V)}\right).$$

Since m preserves G-invariants (because G-invariants can be written as the intersection of equalizers), we see that

$$\left(m\left(\mathbb{F}_p^{\mathrm{Epi}(W,V)}\right)\right)^G \cong m\left(\left(\mathbb{F}_p^{\mathrm{Epi}(W,V)}\right)^G\right).$$

Finally, it is not difficult to show that $\left(\mathbb{F}_p^{\mathrm{Epi}(W,V)}\right)^G = \mathbb{F}_p^{\mathrm{Epi}(W,V)/G}$.

If G is a subgroup of $\mathrm{GL}(V)$, then $\omega_V H^*(V) \subseteq m(\mathbb{F}_p^{\mathrm{Epi}(W,V)})$ implies

$$(\omega_V H^*(V))^G \subseteq \left(m\left(\mathbb{F}_p^{\mathrm{Epi}(W,V)}\right)\right)^G.$$

Since ω_V is defined to be the top Dickson invariant, it is G-invariant for any subgroup G of $\mathrm{GL}(V)$, so the argument in the previous paragraph implies that

$$\begin{aligned}\omega_V H^*(V)^G &\cong (\omega_V H^*(V))^G \\ &\subseteq \left(m\left(\mathbb{F}_p^{\mathrm{Epi}(W,V)}\right)\right)^G \\ &\cong m\left(\mathbb{F}_p^{\mathrm{Epi}(W,V)/G}\right).\end{aligned}$$ □

We must introduce some notation before we state the next proposition. Let $\mathcal{A}_k(P)$ be the full subcategory of Quillen's category $\mathcal{A}(P)$ whose objects are the elementary abelian p-subgroups of P of rank at most k. Then, define $\mathcal{H}^*_{(k)}(P)$ by the following:

$$\mathcal{H}^*_{(k)}(P) := \varprojlim_{\mathcal{A}_k(P)} H^*(V).$$

Proposition 4.7 *With $\mathcal{H}^*_{(k)}(P)$ defined as above, we have*

$$m\left(\mathbb{F}_p^{F_k\mathrm{Rep}(W,P)}\right) = \mathcal{H}^*_{(k)}(P).$$

Remark 4.8 When $k = \mathrm{r}_p(P)$ in Proposition 4.7, we see that

$$m\left(\mathbb{F}_p^{\mathrm{Rep}(W,P)}\right) = \mathcal{H}^*(P).$$

This special case of the proposition is clear from [19].

Proof: This proof consists of one major claim.

Claim We can describe $F_k\mathrm{Rep}(W, P)$ in this way:

$$F_k\mathrm{Rep}(W, P) = \operatorname*{colim}_{V \in \mathcal{A}_k(P)} \mathrm{Hom}(W, V).$$

Proof of Claim: We will verify this equality by showing that $F_k\mathrm{Rep}(W, P)$ has the properties of the colimit. We must first show that for every V in $\mathcal{A}_k(P)$ there is a map $j_V : \mathrm{Hom}(W, V) \to F_k\mathrm{Rep}(W, P)$. This map is easy to define; it sends $f \mapsto [f]$, the class of f in $F_k\mathrm{Rep}(W, P)$. We must now prove that these maps are compatible; in other words, we must show that given $\alpha : V_1 \to V_2$ in $\mathcal{A}_k(P)$, we have $j_{V_1} = j_{V_2} \circ \alpha_*$. This is equivalent to showing that $[f] = [\alpha \circ f]$ in $F_k\mathrm{Rep}(W, P)$. Since α is a map in $\mathcal{A}_k(P)$, there exists an element $p \in P$ such that α equals c_p restricted to V_1. Therefore, since $\alpha \circ f = c_p \circ f$, it is easy to see that $[f] = [\alpha \circ f]$.

We also need to show that $F_k\mathrm{Rep}(W, P)$ has the required universal property. Assume that there is another set D with compatible maps $f_i : \mathrm{Hom}(W, V_i) \to D$ for every V_i. We must describe a map $\phi : F_k\mathrm{Rep}(W, P) \to D$ such that $\phi \circ j_{V_i} = f_i$ for every V_i. Let $[g]$ be an element in $F_k\mathrm{Rep}(W, P)$; if the image of g is V_i, then define $\phi([g])$ by $\phi([g]) := f_i(g)$. We must verify that ϕ described in this way is well-defined. If $[g_1] = [g_2]$ in $F_k\mathrm{Rep}(W, P)$, there is an element $p \in P$ such that $g_2 = c_p \circ g_1$. Let the images of the maps g_1 and g_2 be V_1 and V_2 respectively, and let α be defined to be $c_p|_{V_1}$. It is clear that α is a map $V_1 \to V_2$ in $\mathcal{A}_k(P)$. For ϕ to be well-defined, we would need to have $f_1(g_1) = f_2(g_2)$, but since $g_2 = \alpha \circ g_1$, this was part of our assumptions about D.

We must finally prove that ϕ is unique. Suppose that we are given another map $\gamma : F_k\mathrm{Rep}(W, P) \to D$ such that $f_i = \gamma \circ j_{V_i}$ for every V_i; we must show that $\gamma = \phi$. If $[g]$ is an element of $F_k\mathrm{Rep}(W, P)$, let the image of g be V_0. By our definition of j_{V_0}, we know that $[g] = j_{V_0}(g)$ for $g \in \mathrm{Hom}(W, V_0)$. Then

$$\begin{aligned}\gamma([g]) &= \gamma(j_{V_0}(g)) \\ &= f_0(g) \\ &= \phi([g]),\end{aligned}$$

which verifies the claim. $\Diamond$

The claim now serves as the linchpin for the proof of Proposition 4.7. For now,

$$\begin{aligned}\mathbb{F}_p^{F_k\mathrm{Rep}(W,P)} &= \mathbb{F}_p^{\mathrm{colim}_{V\in\mathcal{A}_k(P)}\mathrm{Hom}(W,V)} \\ &= \varprojlim_{\mathcal{A}_k(P)} \mathbb{F}_p^{\mathrm{Hom}(W,V)}.\end{aligned}$$

Therefore, since m preserves limits, when we apply m we have

$$\begin{aligned} m\left(\mathbb{F}_p^{F_k \mathrm{Rep}(W,P)}\right) &= m\Big(\lim_{\substack{\longleftarrow \\ \mathcal{A}_k(P)}} \mathbb{F}_p^{\mathrm{Hom}(W,V)}\Big) \\ &= \lim_{\substack{\longleftarrow \\ \mathcal{A}_k(P)}} m\left(\mathbb{F}_p^{\mathrm{Hom}(W,V)}\right) \\ &= \lim_{\substack{\longleftarrow \\ \mathcal{A}_k(P)}} H^*(V) \\ &= \mathcal{H}^*_{(k)}(P). \end{aligned}$$

□

Our attention now turns explicitly to the calculation of dimension.

Proposition 4.9 *Let S be a simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module, let e_S be the corresponding idempotent, and let $\mathrm{rk}(V) = k$. If $e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$, we have*

$$\dim(e_S\mathcal{H}^*_{(k)}(P)) \geq k.$$

Proof: If we put together (4.8) and Propositions 4.6 and 4.7, we see that

$$\omega_V H^*(V)^{\mathrm{GL}(V)_P} \otimes Z \subseteq e_S\mathcal{H}^*_{(k)}(P), \tag{4.11}$$

for a nonzero module Z. The functor m has the property that $m(F \otimes Y) = m(F) \otimes Y$ for all functors $F \in \mathcal{F}$ when Y is a fixed $\mathbb{Z}/p$-vector space. In our situation, therefore,

$Z = e_S \mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V}$. Now, since $\mathrm{GL}(V)_P \leq \mathrm{GL}(V)$, we have

$$H^*(V)^{\mathrm{GL}(V)} \subseteq H^*(V)^{\mathrm{GL}(V)_P}$$

and consequently

$$\omega_V H^*(V)^{\mathrm{GL}(V)} \subseteq \omega_V H^*(V)^{\mathrm{GL}(V)_P}.$$

When $p = 2$, it is a classical result that $H^*(V)^{\mathrm{GL}(V)}$ is a polynomial algebra on k variables (see [35]). If $p > 2$, it is noted on [7, p.598] that the polynomial part of $H^*(V)^{\mathrm{GL}(V)}$, denoted $P(V)^{\mathrm{GL}(V)}$, has this same structure. In either case, because of the definition of ω_V, there is a polynomial algebra of dimension k inside $\omega_V H^*(V)^{\mathrm{GL}(V)}$. Thus, $\dim(\omega_V H^*(V)^{\mathrm{GL}(V)}) \geq k$, meaning that (4.11) implies

$$\dim(e_S \mathcal{H}^*_{(k)}(P)) \geq k. \qquad \square$$

The final ingredient in the proof of Theorem 4.5 is the following proposition.

Proposition 4.10 *Let S be a simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module, let e_S be the corresponding idempotent, and let $\mathrm{rk}(V) = k$. If $e_S \mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$, we have*

$$\dim(e_S \mathcal{H}^*(P)) \geq k.$$

Proof: Proposition 4.7 shows that the epimorphism in (4.9) turns into the following $\mathcal{N}il$-epimorphism after we apply m:

$$e_S\mathcal{H}^*(P) \to e_S\mathcal{H}^*_{(k)}(P).$$

We now appeal to Proposition 3.20. Since $e_S\mathcal{H}^*_{(k)}(P)$ is clearly reduced (because it is in the image of m), we only need to verify that $e_S\mathcal{H}^*(P)$ and $e_S\mathcal{H}^*_{(k)}(P)$ are objects in $K_{fg}-\mathcal{U}$ for some appropriate K. It is a classic fact that $H^*(P)$ is noetherian (see [2, p.143]). We also know that if $V \leq P$, $H^*(V)$ is a finitely generated $H^*(P)$-module via the restriction map (see [14, p.236]). Now, since

$$\mathcal{H}^*_{(k)}(P) = \lim_{\substack{\longleftarrow \\ \mathcal{A}_k(P)}} H^*(V) \subseteq \prod_{\mathcal{A}(P)} H^*(V),$$

and since $\prod H^*(V)$ is finitely generated over $H^*(P)$, we can see that $\mathcal{H}^*_{(k)}(P)$ is finitely generated over $H^*(P)$ for any k. (Here we also use the fact that $\mathcal{H}^*_{(k)}(P)$ is an $H^*(P)$-submodule of $\prod H^*(V)$.) This shows that both $\mathcal{H}^*_{(k)}(P)$ and $\mathcal{H}^*(P)$ are in $H^*(P)_{fg}-\mathcal{U}$. Proposition 3.20 gives $\dim(e_S\mathcal{H}^*(P)) \geq \dim(e_S\mathcal{H}^*_{(k)}(P))$, but since we showed in Proposition 4.9 that $\dim(e_S\mathcal{H}^*_{(k)}(P)) \geq k$, we have $\dim(e_S\mathcal{H}^*(P)) \geq k$. $\square$

We now have all of the tools necessary to prove Theorem 4.5.

Proof (of Theorem 4.5): We know that $e_S\mathcal{H}^*(P) = \overline{R}_0(e_SH^*(P))$. Further, Theorem 3.21 shows that $\dim(e_SH^*(P)) \geq \dim(\overline{R}_s(e_SH^*(P))$ for every s. Therefore, since $\dim(e_S\mathcal{H}^*(P)) \geq \mathrm{rk}(V)$, we may conclude that $\dim(e_S H^*(P)) \geq \mathrm{rk}(V)$, as desired. □

We now look to combine Theorem 4.5 with the discussion in Section 4.1. To do so, we first define a number related to the rank of an elementary abelian p-subgroup of a group G.

Definition 4.11 Let G be a finite p-group and let U and V be elementary abelian p-subgroups of G. The *maximal rank* of V in G, denoted $\mathrm{mrk}_G(V)$, is defined in the following way:

$$\mathrm{mrk}_G(V) := \max\{\mathrm{rk}(U) \mid V \leq U\}.$$

Theorem 4.12 *Let S be a simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module and let e_S be the corresponding idempotent. If $e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$, we have $\dim(e_S H^*(P)) \geq \mathrm{mrk}_P(V)$.*

We have an immediate corollary to this theorem, which appears as Proposition 4.2 in [26].

Corollary 4.13 (Martino-Priddy) *If $e\mathcal{H}^*(P) \neq 0$, then $\dim(e\mathcal{H}^*(P))$ is at least as big as the smallest rank of a maximal elementary abelian p-subgroup of P.*

Proof: If $e\mathcal{H}^*(P) \neq 0$, we know that $e\mathbb{F}_p^{\mathrm{Rep}(W,P)} \neq 0$, and thus by (4.4) we have $e\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$ for some V. Then by the proof of Theorem 4.12 we see that $\dim(e\mathcal{H}^*(P)) \geq \mathrm{mrk}_P(V)$. □

We have now laid the foundation for calculating the size of $\dim(e_S H^*(P))$. In the following section, we describe one set of conditions under which we can answer Question 1.9.

4.3 An answer for certain groups

We wish to develop the remaining theory necessary to determine for which groups we can solve our dissertation problem. We will begin with some group theoretic results.[1]

Lemma 4.14 *Suppose G is a finite group and that G_0 is a p-subgroup of G. Let M be a simple $\mathbb{F}_p[G]$-module. Then M is a composition factor of $(1_{G_0})^G$, where $(1_{G_0})^G$ denotes the trivial G_0-module extended to G.*

Proof: We first claim that since G_0 is a p-group, there exists a nonzero $x \in M$ such that G_0 fixes x. Choose a nonzero element y of M and examine the abelian subgroup of M generated by the set $\{g \cdot y \,|\, g \in G_0\}$. This is a finite abelian p-group with an obvious G_0-action. Since all orbits must have order a power of p, the number of fixed points must also be divisible by p. Since 0 must be fixed, there must be at least one nonzero fixed point. This is our element x.

[1]Lemmas 4.14 and 4.17 in this section were discussed in a private communication between J. Thompson and N. Kuhn in December, 1988. The author wishes to express his thanks to N. Kuhn for making this communication available.

We may describe the induced module in question in the following way:

$$(1_{G_0})^G = \mathbb{F}_p[G] \otimes_{\mathbb{F}_p[G_0]} 1_{G_0}.$$

Therefore, we can map $(1_{G_0})^G \to M$ by $g \otimes 1 \mapsto gx$. Since M is irreducible and since G_0 fixes the nonzero element x, this map is a surjection. This shows that M must be a composition factor of $(1_{G_0})^G$. □

Let P be a finite p-group. Let A be a free abelian group of rank $\mathrm{r}_p(P)$, and let $\Phi = \mathrm{Hom}(A, P)$. In what follows we will use the actions of both $\mathrm{Aut}(P)$ and P on Φ.

Definition 4.15 If G is a group and H is a subgroup of G, then we say that H is *self-centralizing* if $C_G(H) = H$.

Remark 4.16 We will denote the set of subgroups of a group G which are both self-centralizing and normal by $\mathrm{SCN}(G)$. It can be shown that if G is a finite p-group, $\mathrm{SCN}(G)$ is non-empty.

Lemma 4.17 *If $\phi \in \Phi$ and $\phi(A) \in \mathrm{SCN}(P)$, then the stabilizer in $\mathrm{Aut}(P)$ of the P-orbit of ϕ is a p-group.*

PROOF: We will denote the stabilizer in $\mathrm{Aut}(P)$ of the P-orbit of ϕ by $\mathrm{Aut}(P)_{[\phi]}$. Let $\alpha \in \mathrm{Aut}(P)_{[\phi]}$. Then there exists $g \in P$ such that $c_g\alpha\phi = \phi$. Set $\gamma := c_g\alpha$; since we want to show that α is a p-element, it suffices to show that γ is a p-element. This will prove the lemma.

For each $a \in A$, $\gamma(\phi(a)) = \phi(a)$, so γ is an automorphism of P which fixes each element of $\phi(A)$. Let $\gamma = \gamma_p\gamma_{p'} = \gamma_{p'}\gamma_p$, where γ_p is a p-element and $\gamma_{p'}$ is a p-regular element. We want to show that $\gamma_{p'} = 1$.

We form the semi-direct product of $\langle\gamma_{p'}\rangle$ with P and call it P^*. Then

$$\begin{aligned} C_{P^*}(\phi(A)) &= \langle\phi(A), \gamma_{p'}\rangle \\ &= \phi(A) \times \langle\gamma_{p'}\rangle, \end{aligned}$$

where the first equality holds because of the definition of the centralizer and the fact that $\phi(A)$ is self-centralizing, and the second equality holds because $\gamma_{p'}$ fixes each element of $\phi(A)$. Since $\phi(A) \lhd P$, we see that $\phi(A) \lhd P^*$. Then, $\langle\gamma_{p'}\rangle$ is characteristic in $C_{P^*}(\phi(A))$ since it is the unique p'-subgroup of $C_{P^*}(\phi(A))$. Finally, since $\phi(A) \lhd P^*$, we have $C_{P^*}(\phi(A)) \lhd P^*$ and thus $\langle\gamma_{p'}\rangle \lhd P^*$. This means that $P^* = \langle\gamma_{p'}\rangle \times P$, and $\gamma_{p'} = 1$. □

Note that Φ/P is $\mathrm{Rep}(A, P)$ and that $\mathrm{Aut}(P)_{[\phi]}$ can also be written as $\mathrm{Out}(P)_V$. Lemmas 4.14 and 4.17 show that every simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module occurs as a composition factor in $\left(1_{\mathrm{Out}(P)_V}\right)^{\mathrm{Out}(P)} \simeq \mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_V]$ if $V = \phi(A)$ is in $\mathrm{SCN}(P)$.

We now bring together results from throughout this chapter to prove a major theorem.

Theorem 4.18 *Let P be a finite p-group and let V be an elementary abelian p-subgroup of P. If $V \in \mathrm{SCN}(P)$, then $\dim(e_S H^*(P)) \geq \mathrm{mrk}_P(V)$ for every simple*

$\mathbb{F}_p[\mathrm{Out}(P)]$*-module* S*.*

Proof: As noted above, Lemmas 4.14 and 4.17 show that every simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module occurs as a composition factor in $\mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_V]$ if $V \in \mathrm{SCN}(P)$. Since $\mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_V]$ and $\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V}$ are dual modules, $e_S\mathbb{F}_p^{\mathrm{Out}(P)/\mathrm{Out}(P)_V} \neq 0$. Finally, we invoke Theorem 4.12 to conclude that $\dim(e_S H^*(P)) \geq \mathrm{mrk}_P(V)$. □

Corollary 4.19 *Let* P *be a finite* p*-group and let* V *be an elementary abelian* p*-subgroup of* P*. If* $V \in \mathrm{SCN}(P)$*, and* $\mathrm{mrk}_P(V) = \mathrm{r}_p(P)$*, then* $\dim(e_S H^*(P)) = \mathrm{r}_p(P)$ *for every simple* $\mathbb{F}_p[\mathrm{Out}(P)]$*-module* S*.*

Proof: Since we already knew that $\dim(e_S H^*(P)) \leq \mathrm{r}_p(P)$, we can conclude that $\dim(e_S H^*(P)) = \mathrm{r}_p(P)$ from Theorem 4.18 if $\mathrm{mrk}_P(V) = \mathrm{r}_p(P)$. □

Remark 4.20 In the proof of Theorem 4.18, the conditions on the group P were used to show that $\mathrm{Out}(P)_V$ was a p-group. After this, we depended only on Lemmas 4.14 and 4.17. Therefore we remark that Question 1.9 also has an affirmative answer for a group P if P contains an elementary abelian p-subgroup V with $\mathrm{mrk}_P(V) = \mathrm{r}_p(P)$ such that $\mathrm{Out}(P)_V$ is a p-group.

We have discussed some situations in which the answer to Question 1.9 is known, and we have utilized the modules $\overline{R}_0(H^*(P))$ in this process. In the following chapter we will present explicit formulas for calculating these and other objects related to the localization of $\mathcal{U}$ away from $\mathcal{N}il_n$.

Chapter 5

Formulas for $L_n(H^*(G))$, $\overline{R}_n(H^*(G))$ and $\mathrm{LF}(H^*(G))$

We have used the strength of the nilpotent filtration to provide an initial answer to Question 1.9. However, we have not yet presented specific calculations of any localizations. In Section 5.1 we present a description of $L_n(H^*(G))$ from [20] and then derive an expression for $\overline{R}_n(H^*(G))$. In Section 5.2 we define locally finite modules and calculate $\mathrm{LF}(H^*(G))$.

5.1 Localization away from $\mathcal{N}il_n$

Throughout [20], Henn, Lannes, and Schwartz give several descriptions of localization away from $\mathcal{N}il_n$. For our purposes, the most helpful discussion occurs in Section I.5, when the focus is on equivariant group cohomology.

Let E be an elementary abelian p-subgroup of G, and let $C_G(E)$ denote the centralizer of E in G. Let c_E be the following composition:

$$c_E : E \times C_G(E) \xrightarrow{\text{mult}} C_G(E) \xhookrightarrow{\ i\ } G.$$

We note that c_E is a homomorphism precisely because the multiplication is taking place within the centralizer. The next proposition gives our first description of $L_n(H^*(G))$.

Proposition 5.1 (Theorem I.5.5 in [20]) *The map from $H^*(G)$ to*

$$\text{Eq} : \left\{ \prod_{E \in \mathcal{A}(G)} H^*(E) \otimes \left(H^*(C_G(E))\right)^{<n} \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{E_1 \xrightarrow{\alpha} E_2} H^*(E_1) \otimes \left(H^*(E_1 \times C_G(E_2))\right)^{<n} \right\}$$

induced by the maps c_E^* is localization away from $\mathcal{N}il_n$.

In Proposition 5.1, the products are taken over all objects and all morphisms of $\mathcal{A}(G)$, Quillen's category (see Section 2.5). For a map $\alpha : E_1 \to E_2$, both $\mu(\alpha)$ and $\nu(\alpha)$ are induced by group homomorphisms; they correspond (respectively) to the following:

$$\begin{gathered} E_1 \times E_1 \times C_G(E_2) \xrightarrow{\text{mult} \times \text{id}} E_1 \times C_G(E_2) \\ E_1 \times E_1 \times C_G(E_2) \xrightarrow{\text{id} \times \alpha \times \text{id}} E_1 \times E_2 \times C_G(E_2) \xrightarrow{\text{id} \times \text{mult}} E_1 \times C_G(E_2). \end{gathered} \tag{5.1}$$

We will supply an alternate description of $L_n(H^*(G))$ after the following definition.

Definition 5.2 If $\mathcal{C}$ is a category, we define the *twisted arrow category* of $\mathcal{C}$, denoted $\mathcal{C}_\sharp$, in the following way. The objects of $\mathcal{C}_\sharp$ are the maps in $\mathcal{C}$. If f_1 and f_2 are objects in $\mathcal{C}_\sharp$, then a morphism $\alpha : f_1 \to f_2$ consists of a pair of morphisms in $\mathcal{C}$, (g_1, g_2), such that $g_2 f_1 g_1 = f_2$.

The equalizer in Proposition 5.1 can be rewritten as a limit over $\mathcal{A}(G)_\sharp$. We will take this as our working formula for $L_n(H^*(G))$:

$$\lim_{E_1 \xrightarrow{\alpha} E_2} \left[\mathrm{Eq} : \left\{ H^*(E_1) \otimes \Big(H^*(C_G(E_2)) \Big)^{<n} \overset{\mu(\alpha)}{\underset{\nu(\alpha)}{\rightrightarrows}} H^*(E_1) \otimes \Big(H^*(E_1 \times C_G(E_2)) \Big)^{<n} \right\} \right]. \tag{5.2}$$

Remark 5.3 Throughout [20], the authors frequently alternate between descriptions of $L_n(M)$, such as the one in Proposition 5.1 (an equalizer between products) and the one in (5.2). This second representation should be understood as the limit of a functor $\mathcal{A}(G)_\sharp \to \mathcal{U}$. We refer to [20, Section I.1.17] for more details.

In Sections 2.3 and 2.4, we saw that the modules $\overline{R}_n(M)$ are closely related to the localization functors L_n and L_{n+1}. Recall that $\tau_n : L_{n+1} \to L_n$ is a natural transformation and that $\Sigma^s \overline{R}_n(M) = \ker \tau_{n,M}$ by Proposition 2.11. We now use (5.2) to develop a formula for $\overline{R}_n(H^*(G))$.

Proposition 5.4 *The following equalizer is isomorphic to* $\overline{R}_n(H^*(G))$*:*

$$\mathrm{Eq} : \left\{ \prod_{E \in \mathcal{A}(G)} H^*(E) \otimes H^n(C_G(E)) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{E_1 \xrightarrow{\alpha} E_2} H^*(E_1) \otimes H^n(E_1 \times C_G(E_2)) \right\}.$$

Proof: Because the diagrams involved in this argument are large, we introduce some notation:

$$A^{<n+1} := \prod_{E \in \mathcal{A}(G)} H^*(E) \otimes \big(H^*(C_G(E))\big)^{<n+1}$$

$$A^{<n} := \prod_{E \in \mathcal{A}(G)} H^*(E) \otimes \big(H^*(C_G(E))\big)^{<n}$$

$$A^{n} := \prod_{E \in \mathcal{A}(G)} H^*(E) \otimes H^n(C_G(E))$$

$$B^{<n+1} := \prod_{E_1 \xrightarrow{\alpha} E_2} H^*(E_1) \otimes \big(H^*(E_1 \times C_G(E_2))\big)^{<n+1}$$

$$B^{<n} := \prod_{E_1 \xrightarrow{\alpha} E_2} H^*(E_1) \otimes \big(H^*(E_1 \times C_G(E_2))\big)^{<n}$$

$$B^{n} := \prod_{E_1 \xrightarrow{\alpha} E_2} H^*(E_1) \otimes H^n(E_1 \times C_G(E_2)).$$

With these assignments, we can write $L_{n+1}(H^*(G))$ and $L_n(H^*(G))$ in the following way (see Proposition 5.1):

$$L_{n+1}(H^*(G)) = \mathrm{Eq} : \big\{A^{<n+1} \overset{\mu}{\underset{\nu}{\rightrightarrows}} B^{<n+1}\big\}$$

$$L_{n}(H^*(G)) = \mathrm{Eq} : \big\{A^{<n} \overset{\mu}{\underset{\nu}{\rightrightarrows}} B^{<n}\big\}.$$

With this notation in hand, the proof of this proposition is formal. We examine the following diagram, noting that τ_{n,H^*G} is the unique map filling in the dotted-line arrow and making the diagram commute:

$$\begin{array}{ccccc} & & A^n & \overset{\mu}{\underset{\nu}{\rightrightarrows}} & B^n \\ & & \downarrow & & \downarrow \\ L_{n+1}H^*(G) & \hookrightarrow & A^{<n+1} & \overset{\mu}{\underset{\nu}{\rightrightarrows}} & B^{<n+1} \\ \vdots & & \downarrow \pi_1 & & \downarrow \pi_2 \\ L_nH^*(G) & \hookrightarrow & A^{<n} & \overset{\mu}{\underset{\nu}{\rightrightarrows}} & B^{<n}. \end{array}$$

We have denoted the obvious projection maps by π_1 and π_2, and their kernels are clearly the indicated modules. It is then simply a matter of homological algebra to verify that

$$\ker \tau_{n,H^*G} = \mathrm{Eq} : \left\{ A^n \overset{\mu}{\underset{\nu}{\rightrightarrows}} B^n \right\},$$

which proves the proposition. □

Following the discussion in Remark 5.3, we can also write $\overline{R}_n(H^*(G))$ as the following limit over $\mathcal{A}(G)_\sharp$:

$$\lim_{E_1 \overset{\alpha}{\to} E_2} \left[\mathrm{Eq} : \left\{ H^*(E_1) \otimes H^n(C_G(E_2)) \overset{\mu(\alpha)}{\underset{\nu(\alpha)}{\rightrightarrows}} H^*(E_1) \otimes H^n(E_1 \times C_G(E_2)) \right\} \right]. \tag{5.3}$$

A careful examination of $\mu(\alpha)$ and $\nu(\alpha)$ reveals that (5.3) can be adjusted once more. Within the limit, $H^*(E_1)$ can be pulled outside of the equalizer.

Lemma 5.5 *We can calculate $\overline{R}_n(H^*(G))$ as the following limit over $\mathcal{A}(G)_\sharp$:*

$$\lim_{E_1 \xrightarrow{\alpha} E_2} \left[H^*(E_1) \otimes \ \mathrm{Eq} : \left\{ H^n(C_G(E_2)) \underset{\nu(\alpha)'}{\overset{\mu(\alpha)'}{\rightrightarrows}} H^n(E_1 \times C_G(E_2)) \right\} \right].$$

Proof: Recall from (5.1) that both $\mu(\alpha)$ and $\nu(\alpha)$ come from group homomorphisms. Understood in this way, it follows easily that

$$\nu(\alpha) : H^*(E_1) \otimes H^n(C_G(E_2)) \to H^*(E_1) \otimes H^n(E_1 \times C_G(E_2))$$

leaves the first coordinate unchanged. In order to prove the lemma, we must show that $\mu(\alpha)$ does the same. In general, if V is an elementary abelian p-group and if $|x| = n$, the multiplication map in cohomology $H^*(V) \to H^*(V) \otimes H^*(V)$ has the following action:

$$x \longmapsto x \otimes 1 + 1 \otimes x + \sum_{i=1}^{n-1} a_i \otimes b_{n-i}.$$

Here the subscripts on the elements in the sum indicate their dimension in $H^*(V)$. In our situation, since we have the identity map on the $H^n(C_G(E_2))$ factor, dimension constraints ensure that only the first term survives truncation. On a simple tensor, the action of $\mu(\alpha)$ must be $x \otimes y \mapsto x \otimes 1 \otimes y$. Since $\mu(\alpha)$ and $\nu(\alpha)$ agree on $H^*(E_1)$ when considering a simple tensor, this extends to a general element of $H^*(E_1) \otimes H^n(C_G(E_2))$, and our conclusion follows. □

We will now assign some notation to make future discussions of $\overline{R}_n(H^*(G))$ less cumbersome. Let C be a central elementary abelian p-subgroup of G, and define $P_C H^s(G)$ by

$$P_C H^s(G) := \text{ Eq} : \left\{ H^s(G) \overset{m^*}{\underset{p^*}{\rightrightarrows}} H^s(C \times G) \right\},$$

where m and p are (respectively) the multiplication and projection homomorphisms $C \times G \to G$. Since C is central, $H^*(G)$ is an $H^*(C)$-comodule, and $P_C H^*(G)$ represents the $H^*(C)$-primitive elements in $H^*(G)$. We then translate the formula from Lemma 5.5:

$$\overline{R}_n(H^*(G)) = \lim_{\substack{\alpha \in \mathcal{A}(G)_\sharp \\ \alpha : E_1 \to E_2}} \left\{ H^*(E_1) \otimes P_{\alpha(E_1)} H^n(C_G(E_2)) \right\}, \tag{5.4}$$

as $\mathcal{A}$-modules, where we consider $P_{\alpha(E_1)} H^n(C_G(E_2))$ to be in dimension 0.

For a finite group G, we have given explicit descriptions of the modules $L_n(H^*(G))$ and $\overline{R}_n(H^*(G))$. In the following section we will introduce the functor LF and present a formula for $\mathrm{LF}(H^*(G))$.

5.2 Locally finite modules

We begin this section with a brief discussion of locally finite modules. The following definition can be found in [21, p.327].

Definition 5.6 An unstable module M is said to be *locally finite* if every cyclic submodule of M is finite. In other words, M is locally finite if, for every x in M, only a finite number of elements of $\mathcal{A}$ act non-trivially on x.

There are several other characterizations of locally finite modules, some of which involve the functor T_V. We refer the reader to [33, p.144], where the author states a theorem detailing several equivalent definitions.

Let $\mathcal{LF}$ denote the full subcategory of $\mathcal{U}$ consisting of locally finite unstable modules, and let $\mathrm{LF} : \mathcal{U} \to \mathcal{LF}$ be right adjoint to the inclusion $\mathcal{LF} \hookrightarrow \mathcal{U}$. Then $\mathrm{LF}(M)$ is the largest locally finite submodule of M.

Lemma 5.7 *If M is a reduced module, then* $\mathrm{LF}(M) = M^0$.

PROOF: This proof relies on characterizations of reduced modules found in [33, pp.47,48]. If M is a reduced module and $p = 2$, then Sq_0 is injective. If $p > 2$, for every $x \in M$ there exists an element $\theta \in \mathcal{A}$ such that $\mathrm{P}_0^i \theta x \neq 0$ for all i. In both cases, $\mathcal{A}x$ is infinite when $|x| > 0$, and we have proved the lemma. $\square$

The next proposition appears as Proposition 2.11 in [21], and we will use it in the subsequent lemma.

Proposition 5.8 (Proposition 2.11 in [21]) *For all $M \in \mathcal{U}$ and all s,*

$$\mathrm{LF}(M) \cap M^s = \mathrm{nil}_s\, M \cap M^s.$$

There are two immediate consequences of this proposition worth noting. First, $R_s(\mathrm{LF}(M)) = (R_s(M))^0$, or, equivalently, $R_s(\mathrm{LF}(M)) = \mathrm{LF}(R_s(M))$. Additionally, the nilpotent and skeletal filtrations of M agree if and only if M is locally finite.

Lemma 5.9 *For an unstable module M, $(\mathrm{LF}(M))^s = (\overline{R}_s(M))^0$.*

PROOF: We know that $(\mathrm{LF}(M))^s = (\mathrm{nil}_s M)^s$ from Proposition 5.8. Since $\mathrm{nil}_s M$ begins in dimension s, that is, $(\mathrm{nil}_s M)^t = 0$ for $t < s$, we see that

$$\begin{aligned}(\mathrm{nil}_s M)^s &= (\mathrm{nil}_s M / \mathrm{nil}_{s+1} M)^s \\ &= (\Sigma^s R_s(M))^s \\ &= (R_s(M))^0.\end{aligned}$$

Finally, $\mathcal{N}il$-closure does not effect dimension 0, so $(R_s(M))^0 = (\overline{R}_s(M))^0$. □

Proposition 5.10 *The following is an equality of unstable modules:*

$$\mathrm{LF}(H^*(G)) = \lim_{\substack{\alpha \in \mathcal{A}(G)_\sharp \\ \alpha : E_1 \to E_2}} P_{\alpha(E_1)} H^*(C_G(E_2)). \tag{5.5}$$

PROOF: We recall this description of $\overline{R}_n(H^*(G))$ from (5.4):

$$\overline{R}_n(H^*(G)) = \lim_{\substack{\alpha \in \mathcal{A}(G)_\sharp \\ \alpha : E_1 \to E_2}} \left\{ H^*(E_1) \otimes P_{\alpha(E_1)} H^n(C_G(E_2)) \right\}.$$

Lemma 5.9 implies that we only need to consider the 0-dimensional part of $\overline{R}_n(H^*(G))$

to calculate $\mathrm{LF}(H^n(G))$. However, we assume that $P_{\alpha(E_1)}H^n(C_G(E_2))$ is in dimension 0, and since $H^0(E_1) = \mathbb{F}_p$ for all $E_1 \in \mathcal{A}(G)$, we have

$$\begin{aligned}
\mathrm{LF}(H^n(G)) &= \left(\overline{R}_n(H^*(G))\right)^0 \\
&= \left(\lim_{\alpha: E_1 \to E_2} \left\{ H^*(E_1) \otimes P_{\alpha(E_1)} H^n(C_G(E_2)) \right\}\right)^0 \\
&= \lim_{\alpha: E_1 \to E_2} \left\{ \mathbb{F}_p \otimes P_{\alpha(E_1)} H^n(C_G(E_2)) \right\} \\
&= \lim_{\alpha: E_1 \to E_2} P_{\alpha(E_1)} H^n(C_G(E_2)). \qquad \square
\end{aligned}$$

The three major formulas in this chapter ((5.2), (5.4), and (5.5)) have been presented as limits over $\mathcal{A}(G)_\sharp$. In the next chapter we will simplify these expressions by studying a subcategory of $\mathcal{A}(G)_\sharp$.

Chapter 6

Limits over a smaller category

In this chapter we will revisit the formulas for calculating $L_n(H^*(G))$, $\overline{R}_n(H^*(G))$ and $\mathrm{LF}(H^*(G))$ that were given in Chapter 5. We will show that these limits can be taken over a subcategory of $\mathcal{A}(G)_\sharp$ denoted $\mathcal{A}_C(G)_\sharp$. Section 6.1 examines an aspect of group theory that will provide a connection between $\mathcal{A}(G)_\sharp$ and $\mathcal{A}_C(G)_\sharp$. The main theorem of this chapter is found in Section 6.2, where we give an alternate computation of $L_n(H^*(G))$. The expressions for $\overline{R}_n(H^*(G))$ and $\mathrm{LF}(H^*(G))$ are drawn as a corollary.

6.1 The internal direct product construction

Let G be a finite p-group and let C be a central elementary abelian p-subgroup of G. Recall the definition of Quillen's category $\mathcal{A}(G)$ from Section 2.5. We now define the functor $P_C : \mathcal{A}(G) \to \mathcal{A}(G)$; for an object $H \in \mathcal{A}(G)$, $P_C(H)$ is the internal direct product of H with C. That is, $P_C(H) = CH$. Since $C \trianglelefteq G$, CH is a subgroup of G;

it is also an object of $\mathcal{A}(G)$, since it is abelian and every element has order p.

We now specify the action of P_C on morphisms. Recall that for a map $\alpha : H \to H'$ in $\mathcal{A}(G)$, α agrees with conjugation by some $g \in G$ when restricted to H. Then $P_C(\alpha) : CH \to CH'$ sends $ch \mapsto c\,\alpha(h)$. Because $P_C(\alpha)$ also agrees with conjugation by g when restricted to CH, this is still a map in $\mathcal{A}(G)$. It is easy to see that P_C has the proper action on identity and composite maps, so P_C is a functor.

We now define the natural transformation $\gamma : \mathrm{id}_{\mathcal{A}(G)} \to P_C$. Given $H \in \mathcal{A}(G)$, let γ_H be the inclusion $H \hookrightarrow CH$, $h \mapsto 1 \cdot h$. In order to conclude that this is a natural transformation, we need to prove that the following square commutes:

$$\begin{array}{ccc} H_1 & \xrightarrow{\gamma_{H_1}} & CH_1 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle P_C(\alpha)} \\ H_2 & \xrightarrow{\gamma_{H_2}} & CH_2\,. \end{array}$$

This is easy to check on elements:

$$P_C(\alpha) \circ \gamma_{H_1}(h) = P_C(\alpha)(1 \cdot h) = 1 \cdot \alpha(h);$$

$$\gamma_{H_2} \circ \alpha(h) = \gamma_{H_2}(\alpha(h)) = 1 \cdot \alpha(h).$$

We rely upon P_C in the following section to prove the equality of two limits.

6.2 Limits over smaller categories

The module $L_n(H^*(G))$ can be calculated as the limit of a functor, as was made plain in [20] and mentioned in Section 5.1. In this section we state this formula explicitly and show that the limit can be simplified.

For a category $\mathcal{C}$, consider the associated twisted arrow category $\mathcal{C}_\sharp$ (see Section 5.1). Given any object $f \in \mathcal{C}_\sharp$, there are two standard ways to construct a morphism to f in $\mathcal{C}_\sharp$: if $f : A \to B$, then the pair (id_A, f) is a map $\mathrm{id}_A \to f$ and the pair (f, id_B) is a map $\mathrm{id}_B \to f$. We will use these maps later in the proof of Theorem 6.4.

We now describe the category whose twisted arrow category will be our primary focus. Let C denote the largest central elementary abelian p-subgroup of G. We define the category $\mathcal{A}_C(G)$ to be the full subcategory of $\mathcal{A}(G)$ whose objects contain C as a subgroup.

Lemma 6.1 *Let C be defined as above.*

(i) The values of the functor P_C lie in $\mathcal{A}_C(G)$.

(ii) For every $E \in \mathcal{A}(G)$, $C_G(E) = C_G(CE)$.

Proof: It is easy to see that CE contains C as a subgroup, so $CE \in \mathcal{A}_C(G)$. Using the fact that C is central, it is straightforward to check (ii) on elements. □

We now present a simple group theoretic construction. For any map $\alpha : E_1 \to E_2$

in $\mathcal{A}(G)_\sharp$, we define $j_\alpha : C_G(E_2) \to C_G(E_1)$ in the following way. Since the maps in $\mathcal{A}(G)$ are all injective, we know that $E_1 \cong \alpha(E_1)$, and the inclusion $E_1 \cong \alpha(E_1) \hookrightarrow E_2$ gives us

$$j_\alpha : C_G(E_2) \hookrightarrow C_G(\alpha(E_1)) \cong C_G(E_1).$$

The specific action of j_α is easy to describe. Since α agrees with conjugation by some element g of G when restricted to E_1, j_α simply sends $x \mapsto c_g^{-1}(x)$. We now offer two lemmas about these maps.

Lemma 6.2 *For any $E \in \mathcal{A}(G)$, j_{γ_E} is the identity.*

Proof: From Lemma 6.1 we know that $C_G(E) = C_G(CE)$. As a result, j_{γ_E} is the identity, because as a map in $\mathcal{A}(G)$, γ_E corresponds to conjugation by the identity element. Therefore, according to the formula discussed above, $j_{\gamma_E} : x \mapsto c_e^{-1}(x) = x$.□

Lemma 6.3 *For any $\alpha \in \mathcal{A}(G)_\sharp$, the maps j_α and $j_{P_C(\alpha)}$ are identical.*

Proof: Since γ is a natural transformation, the following square commutes for any $\alpha \in \mathcal{A}(G)_\sharp$:

$$\begin{array}{ccc} E_1 & \xrightarrow{\gamma_{E_1}} & CE_1 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle P_C(\alpha)} \\ E_2 & \xrightarrow{\gamma_{E_2}} & CE_2 . \end{array}$$

All of these maps are in $\mathcal{A}(G)_\sharp$, so we can examine the corresponding maps between centralizers. The following square commutes ($j_{\alpha_2\alpha_1} = j_{\alpha_1} j_{\alpha_2}$ is true in general), and the horizontal arrows are the identity by Lemma 6.2:

$$\begin{array}{ccc} C_G(E_1) & \xleftarrow{\text{id}} & C_G(CE_1) \\ \uparrow{\scriptstyle j_\alpha} & & \uparrow{\scriptstyle j_{P_C(\alpha)}} \\ C_G(E_2) & \xleftarrow{\text{id}} & C_G(CE_2). \end{array}$$

This proves the lemma. $\square$

To make this discussion, and the result which follows, more formal, we introduce a functor whose limit calculates $L_n(H^*(G))$ explicitly. Let F be the covariant functor $\mathcal{A}(G)_\sharp \to \mathcal{U}$ which assigns to a map $\alpha : E_1 \to E_2$ the unstable module

$$\text{Eq} : \left\{ H^*(E_1) \otimes \Big(H^*(C_G(E_2))\Big)^{<n} \underset{\nu(\alpha)}{\overset{\mu(\alpha)}{\rightrightarrows}} H^*(E_1) \otimes \Big(H^*(E_1 \times C_G(E_2))\Big)^{<n} \right\}.$$

If $(\beta_1, \beta_2) = \beta : \alpha_1 \to \alpha_2$ is a map in $\mathcal{A}(G)_\sharp$, then $F(\beta) : F(\alpha_1) \to F(\alpha_2)$ is simply $\beta_1^* \otimes j_{\beta_2}^*$. (Checking that $F(\beta)$ lands in $F(\alpha_2)$ is not difficult. As with most verifications in this section, it suffices to show that certain group homomorphisms commute.) This notation means that we can rewrite (5.2) as

$$L_n(H^*(G)) = \lim_{\mathcal{A}(G)_\sharp} F.$$

Since $\mathcal{A}_C(G)_\sharp$ is a subcategory of $\mathcal{A}(G)_\sharp$, we will give the name F_C to the functor which is simply this categorical inclusion followed by F. In order to state the major theorem of this section, we adopt the following notation: let $L := \lim_{\mathcal{A}(G)_\sharp} F$ and $L_C := \lim_{\mathcal{A}_C(G)_\sharp} F_C$.

Theorem 6.4 *The module $L_n(H^*(G))$ can be calculated as a limit over $\mathcal{A}_C(G)_\sharp$. That is, $L \cong L_C$.*

The proof of this theorem will be given in a sequence of lemmas, and the first step is to define a map $\tau : L_C \to L$. Since L is a limit, this can be accomplished formally. For each $\alpha \in \mathcal{A}(G)_\sharp$, we must produce a map $\pi_\alpha : L_C \to F(\alpha)$, and these π_α must be compatible in the following way: for each $\beta : \alpha_1 \to \alpha_2$ in $\mathcal{A}(G)_\sharp$, we must have $\pi_{\alpha_2} = F(\beta)\pi_{\alpha_1}$. Having such compatible maps, the universal property of L states that τ is the unique map with $\phi_\alpha = \pi_\alpha \tau$ for all $\alpha \in \mathcal{A}(G)_\sharp$, where $\phi_\alpha : L \to F(\alpha)$ is the α-component map for L.

Since L_C is a limit, it also has an α-component map, $\psi_\alpha : L_C \to F_C(\alpha)$, for each $\alpha \in \mathcal{A}_C(G)_\sharp$. To define π_α we will use $\psi_{P_C(\alpha)} : L_C \to F_C(P_C(\alpha)) = F(P_C(\alpha))$, and then we will produce a map $f_\alpha : F(P_C(\alpha)) \to F(\alpha)$. It should be noted that f_α cannot originate with a map $P_C(\alpha) \to \alpha$ in $\mathcal{A}(G)_\sharp$ because of the structure of $\mathcal{A}(G)_\sharp$.

Given a map in $\mathcal{A}(G)_\sharp$, $\alpha : E_1 \to E_2$, we now present an explicit description of f_α. Recall the natural transformation $\gamma : \mathrm{id}_{\mathcal{A}(G)} \to P_C$ (discussed in Section 6.1) which gives a map $\gamma_E : E \to CE$ for every $E \in \mathcal{A}(G)$. We define f_α by $f_\alpha := \gamma_{E_1}^* \otimes \mathrm{id}$. The fact that this map lands in $F(\alpha)$ warrants some discussion.

As previously stated, the maps involved in $F(\alpha)$ originate as homomorphisms between groups. We will say that $\mu(\alpha)$ and $\nu(\alpha)$ come (respectively) from the group homomorphisms μ' and ν' and that $\mu(P_C(\alpha))$ and $\nu(P_C(\alpha))$ come (respectively) from μ'' and ν''. To be precise, we have

$$\mu' := (\mathrm{mult} \times \mathrm{id}) : E_1 \times E_1 \times C_G(E_2) \to E_1 \times C_G(E_2)$$

$$\nu' := (\mathrm{id} \times \mathrm{mult}) \circ (\mathrm{id} \times \alpha \times \mathrm{id}) : E_1 \times E_1 \times C_G(E_2) \to E_1 \times C_G(E_2)$$

$$\mu'' := (\mathrm{mult} \times \mathrm{id}) : CE_1 \times CE_1 \times C_G(CE_2) \to CE_1 \times C_G(CE_2)$$

$$\nu'' := (\mathrm{id} \times \mathrm{mult}) \circ (\mathrm{id} \times P_C(\alpha) \times \mathrm{id}) : CE_1 \times CE_1 \times C_G(CE_2) \to CE_1 \times C_G(CE_2).$$

To show that f_α lands in $F(\alpha)$, it will be sufficient to prove that the following diagram commutes when following both possible paths around the square:

$$\begin{array}{ccc} CE_1 \times C_G(CE_2) & \overset{\mu''}{\underset{\nu''}{\leftleftarrows}} & CE_1 \times CE_1 \times C_G(CE_2) \\ \gamma_{E_1} \times \mathrm{id} \uparrow & & \uparrow \gamma_{E_1} \times \gamma_{E_1} \times \mathrm{id} \\ E_1 \times C_G(E_2) & \overset{\mu'}{\underset{\nu'}{\leftleftarrows}} & E_1 \times E_1 \times C_G(E_2) \end{array}.$$

Using the fact that $\gamma : \mathrm{id} \to P_C$ is a natural transformation, this commutativity is easy to check on elements.

This defines $f_\alpha : F(P_C(\alpha)) \to F(\alpha)$ for any $\alpha \in \mathcal{A}(G)_\sharp$, and π_α is defined by $\pi_\alpha := f_\alpha \psi_{P_C(\alpha)}$. We now must show that these maps π_α are compatible in the way previously described. That is, given a map $\beta : \alpha_1 \to \alpha_2$ in $\mathcal{A}(G)_\sharp$, we must show that

$F(\beta)f_{\alpha_1}\psi_{P_C(\alpha_1)} = f_{\alpha_2}\psi_{P_C(\alpha_2)}$. We have the following diagram:

$$\begin{array}{ccccc}
 & \overset{\psi_{P_C(\alpha_1)}}{\nearrow} & F_C(P_C(\alpha_1)) = F(P_C(\alpha_1)) & \xrightarrow{f_{\alpha_1}} & F(\alpha_1) \\
L_C & & \downarrow F(P_C(\beta)) & & \downarrow F(\beta) \\
 & \underset{\psi_{P_C(\alpha_2)}}{\searrow} & F_C(P_C(\alpha_2)) = F(P_C(\alpha_2)) & \xrightarrow[f_{\alpha_2}]{} & F(\alpha_2).
\end{array}$$

The triangle commutes because L_C is a limit, and our task is to verify the commutativity of the square. Suppose that the morphism $\beta = (\beta_1, \beta_2)$ has the following form:

$$\begin{array}{ccc}
E_1 & \xrightarrow{\alpha_1} & E_2 \\
\beta_1 \uparrow & & \downarrow \beta_2 \\
E_3 & \xrightarrow{\alpha_2} & E_4.
\end{array}$$

After sifting through the notation, we see that showing $F(\beta)f_{\alpha_1} = f_{\alpha_2}F(P_C(\beta))$ is equivalent to proving the following lemma.

Lemma 6.5 *We have the desired equality of maps:*

$$(\beta_1^* \otimes j_{\beta_2}^*)(\gamma_{E_1}^* \otimes \mathrm{id}) = (\gamma_{E_3}^* \otimes \mathrm{id})(P_C(\beta_1)^* \otimes j_{P_C(\beta_2)}^*). \tag{6.1}$$

Proof: From Lemma 6.3 we see that we can rewrite the left hand side of (6.1) as $(\beta_1^* \otimes j_{P_C(\beta_2)}^*)(\gamma_{E_1}^* \otimes \mathrm{id})$. We now apply both sides of (6.1) to a simple tensor $x \otimes y$

from $F(P_C(\alpha))$:

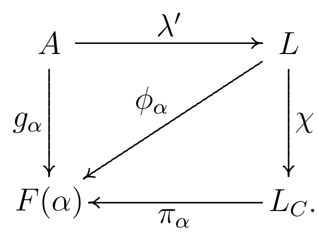
$$\begin{aligned}(\beta_1^* \otimes j^*_{P_C(\beta_2)})(\gamma^*_{E_1} \otimes \mathrm{id})(x \otimes y) &= (\beta_1^* \otimes j^*_{P_C(\beta_2)})(\gamma^*_{E_1}(x) \otimes y) \\ &= \beta_1^* \gamma^*_{E_1}(x) \otimes j^*_{P_C(\beta_2)}(y), \\ (\gamma^*_{E_3} \otimes \mathrm{id})(P_C(\beta_1)^* \otimes j^*_{P_C(\beta_2)})(x \otimes y) &= (\gamma^*_{E_3} \otimes \mathrm{id})(P_C(\beta_1)^*(x) \otimes j^*_{P_C(\beta_2)}(y)) \\ &= \gamma^*_{E_3} P_C(\beta_1)^*(x) \otimes j^*_{P_C(\beta_2)}(y).\end{aligned}$$

Since γ is a natural transformation, we know that $\gamma_{E_1}\beta_1 = P_C(\beta_1)\gamma_{E_3}$ and therefore $\beta_1^*\gamma^*_{E_1} = \gamma^*_{E_3} P_C(\beta_1)^*$. Therefore, the maps in (6.1) agree on any simple tensor $x \otimes y$, and this extends to an arbitrary element of $F(P_C(\alpha))$. □

This completes the definition of τ. We must now show that L_C satisfies the required universal property. Specifically, given any $A \in \mathcal{U}$ and a system of maps $g_\alpha : A \to F(\alpha)$ with the property that $g_{\alpha_2} = F(\beta) g_{\alpha_1}$ for every map $\beta : \alpha_1 \to \alpha_2$ in $\mathcal{A}(G)_\sharp$, there must exist a unique map $\lambda : A \to L_C$ such that $g_\alpha = \pi_\alpha \lambda$ for all $\alpha \in \mathcal{A}(G)_\sharp$.

Assume that we find ourselves in the situation in question. From the information given, the object A and the maps g_α provide a (unique) map $\lambda' : A \to L$ by the universal property of L; this means that $g_\alpha = \phi_\alpha \lambda'$ for each $\alpha \in \mathcal{A}(G)_\sharp$. The obvious map from $L \to L_C$ can be characterized as projection; call this map χ, and define λ to be the composition $\lambda := \chi\lambda'$. As we see from the following diagram, to prove that

$g_\alpha = \pi_\alpha \lambda$ we only need to show that $\phi_\alpha = \pi_\alpha \chi$ for all $\alpha \in \mathcal{A}(G)_\sharp$:



Lemma 6.6 *For any $\alpha \in \mathcal{A}(G)_\sharp$, we have $\phi_\alpha = \pi_\alpha \chi$.*

Proof: Recall that for a map $\beta : \alpha_1 \to \alpha_2$ in $\mathcal{A}(G)_\sharp$ we have $\phi_{\alpha_2} = F(\beta)\phi_{\alpha_1}$ (because L is a limit) and $\pi_{\alpha_2} = F(\beta)\pi_{\alpha_1}$ (this was necessary to define τ). A study of these equalities for three specific maps in $\mathcal{A}(G)_\sharp$ will allow us to prove this lemma. Consider the following objects of $\mathcal{A}(G)_\sharp$: $\alpha : E_1 \to E_2$ and $\gamma_{E_1} : E_1 \to CE_1$.

Fact 6.7 From the map $\beta_1 : \mathrm{id}_{CE_1} \to \gamma_{E_1}$, we see that $\phi_{\gamma_{E_1}} = F(\beta_1)\phi_{\mathrm{id}_{CE_1}}$.

Fact 6.8 From the map $\beta_2 : \mathrm{id}_{E_1} \to \gamma_{E_1}$, we see that $\phi_{\gamma_{E_1}} = F(\beta_2)\phi_{\mathrm{id}_{E_1}}$.

Fact 6.9 From the map $\beta_3 : \mathrm{id}_{E_1} \to \alpha$, we see both that $\phi_\alpha = F(\beta_3)\phi_{\mathrm{id}_{E_1}}$ and $\pi_\alpha = F(\beta_3)\pi_{\mathrm{id}_{E_1}}$.

If we examine Fact 6.9, we see that proving $\phi_\alpha = \pi_\alpha \chi$ is equivalent to checking $F(\beta_3)\phi_{\mathrm{id}_{E_1}} = F(\beta_3)\pi_{\mathrm{id}_{E_1}}\chi$, so it will suffice to show

$$\phi_{\mathrm{id}_{E_1}} = \pi_{\mathrm{id}_{E_1}}\chi. \tag{6.2}$$

The nature of χ guarantees that $\psi_{P_C(\omega)}\chi = \phi_{P_C(\omega)}$ for any $\omega \in \mathcal{A}(G)_\sharp$. This means that since $\pi_\alpha = f_\alpha \psi_{P_C(\alpha)}$, proving (6.2) is the same as showing

$$\phi_{\mathrm{id}_{E_1}} = f_{\mathrm{id}_{E_1}} \phi_{\mathrm{id}_{CE_1}}. \tag{6.3}$$

Facts 6.7 and 6.8 combine to give $\phi_{\mathrm{id}_{E_1}} = F(\beta_1)\phi_{\mathrm{id}_{CE_1}}$, when we add that $F(\beta_2) = \mathrm{id}$. This means we can prove (6.3) by showing $f_{\mathrm{id}_{E_1}} = F(\beta_1)$, but this is immediate since both of these maps are defined to be $\gamma^*_{E_1} \otimes \mathrm{id}$. □

Proof (of Theorem 6.4): To complete the argument that $L \cong L_C$, we must show that λ is unique. Suppose that there is another map $\overline{\lambda} : A \to L_C$ such that $g_\alpha = \pi_\alpha \overline{\lambda}$ for all $\alpha \in \mathcal{A}(G)_\sharp$. We know that $\pi_\alpha = \phi_\alpha \tau$ for all α and also that $\lambda' : A \to L$ is the unique map such that $g_\alpha = \phi_\alpha \lambda'$ for all α. Then, we have $g_\alpha = \pi_\alpha \overline{\lambda} = \phi_\alpha \tau \overline{\lambda}$, meaning that $\tau\overline{\lambda} = \lambda'$ by the uniqueness of λ'. It is clear that $\chi\tau = \mathrm{id}$, since L and L_C agree on $\mathcal{A}_C(G)_\sharp$. Therefore, $\chi\tau\overline{\lambda} = \chi\lambda'$ implies that $\overline{\lambda} = \chi\lambda'$. And, since we defined λ by $\lambda = \chi\lambda'$, this means that $\overline{\lambda} = \lambda$. □

We have shown that $L_n(H^*(G))$ can be calculated in a simpler way than originally presented. Since the description of $\overline{R}_n(H^*(G))$ in (5.4) and $\mathrm{LF}(H^*(G))$ in (5.5) depended on our formula for $L_n(H^*(G))$, Theorem 6.4 means that we also have new descriptions for both of these modules.

Corollary 6.10 *We have the following equalities of unstable modules:*

$$\overline{R}_n(H^*(G)) = \lim_{\alpha \in \mathcal{A}_C(G)_\sharp} \left\{ H^*(E_1) \otimes P_{\alpha(E_1)} H^n(C_G(E_2)) \right\}$$

$$\mathrm{LF}(H^*(G)) = \lim_{\alpha \in \mathcal{A}_C(G)_\sharp} P_{\alpha(E_1)} H^*(C_G(E_2)).$$

In this chapter we have simplified the calculations of Chapter 5 by taking limits over a smaller category. We will reexamine these formulas for particular groups in the following chapter, and this will lead to another answer to Question 1.9.

Chapter 7

p-central groups

In the previous chapter, we replaced $\mathcal{A}(G)_\sharp$ with $\mathcal{A}_C(G)_\sharp$ in several formulas related to $H^*(G)$, and since $\mathcal{A}_C(G)$ is usually a smaller category than $\mathcal{A}(G)$, this reduction made these computations easier. In this chapter we will look at groups G whose structure simplifies these descriptions even further. Section 7.1 introduces these groups and explains these streamlined calculations. We then reconsider Question 1.9 for such groups in Section 7.2.

7.1 Revisiting formulas

In this section we will study specific groups for which the formulas of Section 6.2 are significantly clarified. We will be concerned with groups of the following type.

Definition 7.1 If G is a group in which all elements of order p are central, we say that G is a *p-central group.*

We note that p-central groups have recently proven to be a fruitful area of research. Among other papers, these groups occupy a crucial role in [1] and [34].

If G is a p-central group, it contains a unique maximal elementary abelian p-subgroup C. In the language of Chapter 6, this means that the category $\mathcal{A}_C(G)$ consists of only one object. Limits over $\mathcal{A}_C(G)_\sharp$ become quite simple in this case, as Henn, Lannes, and Schwartz point out when calculating $L_n(H^*(G))$.

Proposition 7.2 *Let G be a p-central group. Then the following are equalities of unstable modules:*

1. $L_n(H^*(G)) = \mathrm{Eq} : \left\{ H^*(C) \otimes \left(H^*(G)\right)^{<n} \overset{\mu}{\underset{\nu}{\rightrightarrows}} H^*(C) \otimes \left(H^*(C) \otimes H^*(G)\right)^{<n} \right\}$;
2. $\overline{R}_s(H^*(G)) = H^*(C) \otimes P_C H^s(G)$; *and*
3. $\mathrm{LF}(H^*(G)) = P_C H^*(G)$.

Remark 7.3 The first item in this proposition appears as [20, Corollary I.5.9]. The last two items are an application of Corollary 6.10. Recall that we consider $P_C H^s(G)$ to be in degree zero.

The expression for $\overline{R}_n(H^*(G))$ in Proposition 7.2 allows us to successfully revisit our dimension question.

7.2 An answer for p-central groups

The main problem we have been studying in this dissertation concerns the dimension of $e_S H^*(G)$, where S is a simple $\mathbb{F}_p[\mathrm{Out}(G)]$-module. We proved in Theorem 3.21

that $\dim(e_S H^*(G)) = \max\{\dim(e_S \overline{R}_n(H^*(G)))\}$, and in Chapter 4 we concluded that $\dim(e_S H^*(G)) = \mathrm{r}_p(G)$ for certain groups by showing $\dim(e_S \overline{R}_0(H^*(G))) = \mathrm{r}_p(G)$. We can give an answer to Question 1.9 for p-central groups, but to do so we may have to venture further into the nilpotent filtration.

Theorem 7.4 *If G is a p-central group, then $\dim(e_S H^*(G)) = \mathrm{r}_p(G)$ for every simple $\mathbb{F}_p[\mathrm{Out}(G)]$-module S.*

PROOF: Since $e_S H^*(G) \neq 0$, we must have $e_S \overline{R}_n(H^*(G)) \neq 0$ for some n; we use the description of $\overline{R}_n(H^*(G))$ from Proposition 7.2. Let $P(C)$ be the polynomial part of $H^*(C)$ (see Section 3.5). It is clear that $\mathrm{Out}(G)$ acts on $H^*(C)$ via the obvious homomorphism $\mathrm{Out}(G) \to \mathrm{GL}(C)$, and we have $P(C)^{\mathrm{GL}(C)} \subseteq H^*(C)^{\mathrm{Out}(G)}$. Now, $H^*(C)^{\mathrm{Out}(G)}$ acts on $H^*(C) \otimes P_C H^n(G)$ on the left, and this commutes with the diagonal $\mathbb{F}_p[\mathrm{Out}(G)]$-action on $H^*(C) \otimes P_C H^n(G)$. Since the action of $H^*(C)^{\mathrm{Out}(G)}$ is free, there is a copy of $P(C)^{\mathrm{GL}(C)}$ within $e_S \overline{R}_n(H^*(G))$, and since $\dim(P(C)^{\mathrm{GL}(C)}) = \mathrm{rk}(C)$, $\dim(e_s \overline{R}_n(H^*(G))) \geq \mathrm{rk}(C)$. Now $\mathrm{rk}(C) = \mathrm{r}_p(G)$ since G is a p-central group, meaning that $\dim(e_S H^*(G)) = \mathrm{r}_p(G)$. □

We will give a second proof of this theorem after we introduce one piece of terminology. The *depth* of a ring R is the length of its longest regular sequence. (See [5, Definition 5.4.9] for more details.) This number is always less than or equal to $\dim(R)$. If the depth and Krull dimension of a ring are equal, we say that the ring is *Cohen-Macaulay.* We will deduce Theorem 7.4 as a corollary of the following theorem.

Theorem 7.5 *If G is a p-group, then $\dim(e_S H^*(G))$ is at least as big as the p-rank of the center of G.*

Proof: We assume that $|G| = p^r$, $n = \mathrm{r}_p(G)$, and $s = r - n$. Let d denote the depth of the ring $H^*(G)$. Let $w_1, \ldots, w_d$ be the bottom d Dickson invariants in $D(G)$ (see Section 3.5); these are the elements which restrict to $\{(v_{p^r - p^i})^{p^s} \mid i = n-d, \ldots, n-1\}$ on an elementary abelian p-subgroup of G of maximal dimension.

We now invoke [7, Proposition 5.2]. There is an embedding

$$\phi : D_{n,s} = \mathbb{F}_p[c_1^{p^s}, \ldots, c_n^{p^s}] \to D(G)$$

such that $D(G)$ is a finitely generated $D_{n,s}$-module via ϕ. This proposition then says that the depth of $H^*(G)$ is the largest r such that $\{(c_1)^{p^s}, \ldots, (c_r)^{p^s}\}$ is a regular sequence on $H^*(G)$. We see, therefore, that $w_1, w_2, \ldots, w_d$ is a regular sequence on $H^*(G)$. Since the elements of $D(G)$ are $\mathrm{Out}(G)$-invariant, $e_S H^*(G)$ is free over $\mathbb{F}_p[w_1, \ldots, w_d]$, meaning that $\dim(e_S(H^*(G))) \geq d$.

Finally, a theorem of J. Duflot [13] shows that the depth of $H^*(G)$ must be greater than or equal to the rank of the center of G. This proves the theorem. □

Remark 7.6 In [26, Proposition 4.1], Martino and Priddy show that any superdominant summand of $\Sigma^\infty BG$ must have dimension greater than or equal to the rank of the center of G, and we have essentially reproduced their proof. They refer to a preprint of D. Rusin, which was never published. We cite the result of Bourguiba and

Zarati at the same point in the argument.

The proof of Theorem 7.5 shows that if $H^*(G)$ is Cohen-Macaulay, Question 1.9 has a positive answer. We should mention, however, that there is no group theoretic condition which ensures that $H^*(G)$ is Cohen-Macaulay.

Corollary 7.7 *If G is a p-central group, then* $\dim(e_S H^*(G)) = \mathrm{r}_p(G)$.

PROOF: Since $\mathrm{rk}(Z(G)) = \mathrm{r}_p(G)$ when G is a p-central group, this follows immediately from the theorem. □

In this chapter we have reexamined some of our earlier theory in the case of p-central groups. These groups also frequently lend themselves to calculations. We will present several such examples, along with other computations, in the following chapter.

Chapter 8

Examples

In this final chapter, we will apply some of our theory to specific 2-groups. In Section 8.1, we will study a spectral sequence calculation for a certain 2-central group of order 32. Then in Section 8.2, we examine the subquotients in the nilpotent filtration carefully for two specific 2-groups. We will then survey all "small" 2-groups in Section 8.3, and we will count the number of these groups for which Question 1.9 is answered. Throughout this chapter we follow the convention of labelling a group with its Hall-Senior number (see [16]).

8.1 Using a spectral sequence

Since p-central groups can be realized as central extensions, we can use the Lyndon-Hochshield-Serre spectral sequence to calculate their cohomology. In this section, we will focus our attention on the 2-central group 32#18. (This is group $W(2)$ in the terminology of [1].) The cohomology of this group, and all other groups of order 32, can be found in [31].

This group fits into the following central extension:

$$1 \to (\mathbb{Z}/2)^3 \to G \to (\mathbb{Z}/2)^2 \to 1.$$

We label the center of this group $C := (\mathbb{Z}/2)^3$ so that $G/C \cong (\mathbb{Z}/2)^2$. The E_2 page of the spectral sequence is $H^*(C) \otimes H^*(G/C)$, so we let $H^*(C) = \mathbb{F}_2[a, b, c]$ and $H^*(G/C) = \mathbb{F}_2[x, y]$, where all of the generators are in dimension 1. Then the differential d_2 has the following action:

$$a \mapsto x^2 \qquad b \mapsto xy \qquad c \mapsto y^2$$
$$x \mapsto 0 \qquad y \mapsto 0.$$

If we let $u = ay + bx$ and $v = by + cx$, we can show that $E_3^{*,*}$ has the form

$$\mathbb{F}_2[a^2, b^2, c^2, x, y, u, v]/(x^2, xy, y^2, u^2, v^2, ux, vy, uv, uy + vx).$$

It is easy to see that the differential d_3 is zero, meaning $E_3^{*,*} = E_\infty^{*,*}$. We can also describe this cohomology ring as $H^*(G) \cong \mathbb{F}_2[a^2, b^2, c^2] \otimes P$, where P is the Poincaré duality algebra $\langle 1, x, y, u, v, uy \rangle$. Algebras such as these are studied in [6].

It is known that $H^*(G)$ is an $H^*(C)$-comodule, and for p-central groups, we can characterize $\mathrm{LF}(H^*(G))$ as the collection of $H^*(C)$-primitive elements. This group is particularly interesting to study because it is the first one in which the primitive elements are not simply the image of the inflation map from $H^*(G/C)$. One can show that the $H^*(C)$-primitive elements are $\{1, x, y, uy\}$. In other words, it is easy to check that u and v are not primitive elements.

8.2 Calculations of $R_n(H^*(P))$ and $\overline{R}_n(H^*(P))$

We have depended on the modules $R_n(H^*(P))$ and $\overline{R}_n(H^*(P))$ for many of the results in this dissertation, and in this section we present a list of these modules for two p-central groups. Both of these examples illustrate why we prefer to work with $\overline{R}_n(H^*(P))$ instead of $R_n(H^*(P))$.

Example 8.1 We first examine the group Q_8. It fits into the central extension

$$1 \to \mathbb{Z}/2 \to Q_8 \to (\mathbb{Z}/2)^2 \to 1,$$

so its cohomology can also be calculated using the Lyndon-Hochshield-Serre spectral sequence. If we use the notation $\mathbb{F}_2[a]$ for $H^*(Z(Q_8))$ and $\mathbb{F}_2[x, y]$ for $H^*(Q_8/Z(Q_8))$, with all of the generators in dimension 1, the result is

$$H^*(Q_8; \mathbb{F}_2) \cong \mathbb{F}_2[x, y]/(x^2 + xy + y^2, y^3) \otimes \mathbb{F}_2[a^4].$$

In this case, $\mathrm{LF}(H^*(Q_8))$ is just the image of the inflation map. We can read off the following calculations:

$$R_0(H^*(Q_8)) = \mathbb{F}_2[a^4] \qquad \overline{R}_0(H^*(Q_8)) = \mathbb{F}_2[a]$$

$$R_1(H^*(Q_8)) = \mathbb{F}_2[a^4] \otimes \langle x, y\rangle \qquad \overline{R}_1(H^*(Q_8)) = \mathbb{F}_2[a] \otimes \langle x, y\rangle$$

$$R_2(H^*(Q_8)) = \mathbb{F}_2[a^4] \otimes \langle x^2, y^2\rangle \qquad \overline{R}_2(H^*(Q_8)) = \mathbb{F}_2[a] \otimes \langle x^2, y^2\rangle$$

$$R_3(H^*(Q_8)) = \mathbb{F}_2[a^4] \otimes \langle x^2 y\rangle \qquad \overline{R}_3(H^*(Q_8)) = \mathbb{F}_2[a] \otimes \langle x^2 y\rangle.$$

We note that this is the smallest group P for which all simple $\mathrm{Out}(P)$-modules do not appear immediately in $\overline{R}_0(H^*(P))$. In this case, $\mathrm{Out}(Q_8) = \mathrm{GL}_2(\mathbb{F}_2)$, and it is well known that there are two simple modules for $\mathbb{F}_2[\mathrm{GL}_2(\mathbb{F}_2)]$. The 2-dimensional simple module does not show up in the nilpotent filtration of $H^*(Q_8)$ until $\overline{R}_1(H^*(Q_8))$.

Example 8.2 We wish to reexamine group 32#18 from Section 8.1. This example shows that the modules $R_n(H^*(P))$ and $\overline{R}_n(H^*(P))$ can have unpredictable behavior. In particular, for this group both $R_2(H^*(P))$ and $\overline{R}_2(H^*(P))$ are zero. This is due to the structure of the primitive elements. Here are the calculations:

$$R_0(H^*(G)) = \mathbb{F}_2[a^2, b^2, c^2] \qquad \overline{R}_0(H^*(G)) = \mathbb{F}_2[a, b, c]$$

$$R_1(H^*(G)) = \mathbb{F}_2[a^2, b^2, c^2] \otimes \langle x, y, u, v\rangle \qquad \overline{R}_1(H^*(G)) = \mathbb{F}_2[a, b, c] \otimes \langle x, y, u, v\rangle$$

$$R_2(H^*(G)) = 0 \qquad \overline{R}_2(H^*(G)) = 0$$

$$R_3(H^*(G)) = \mathbb{F}_2[a^2, b^2, c^2] \otimes \langle uy\rangle \qquad \overline{R}_3(H^*(G)) = \mathbb{F}_2[a, b, c] \otimes \langle uy\rangle.$$

This brief section has examined two particular p-central group, for which we already have an answer to Question 1.9 by Theorem 7.4. In the following section we check our progress toward answering this questions for other groups of small order.

8.3 Progress for small 2-groups

Throughout this dissertation, we have compiled several conditions which ensure a positive answer to Question 1.9. We list them here, for the convenience of the reader.

Theorem 8.3 *If G is a p-group, then $\dim(e_S H^*(G)) = \mathrm{r}_p(G)$ for every simple $\mathbb{F}_p[\mathrm{Out}(G)]$-module S if:*

1. *G is a p-central group (see Theorem 7.4);*

2. *$H^*(G)$ is a Cohen-Macaulay ring (see Remark 7.6);*

3. *G contains a self-centralizing, normal elementary abelian p-subgroup V with $\mathrm{mrk}_G(V) = \mathrm{r}_p(G)$ (see Corollary 4.19); and*

4. *G contains an elementary abelian p-subgroup V with $\mathrm{mrk}_G(V) = \mathrm{r}_p(G)$ such that $\mathrm{Out}(G)_V$ is a p-group (see Remark 4.20).*

The mod 2-cohomology calculations for 2-groups of order dividing 64 are readily available. (See [10] or the appendix in [11].) Using these computations, we can determine how many of these groups meet the criteria of Theorem 8.3. In the table that follows, we list the relevant information about several 2-groups and their cohomology rings. We only include groups for which Question 1.9 is interesting; that is, we do not include abelian groups or groups whose outer automorphism group is a 2-group. The last two columns of this table explain (respectively) whether or not G satisfies condition 2 or 3 of Theorem 8.3.

Table 8.1: Groups of order dividing 64

G	$\lvert\mathrm{Out}(G)\rvert$	$\mathrm{r}_2(G)$	p-central group	$H^*(G)$ C-M	$V_{\max} \in \mathrm{SCN}(G)$
8#5	$2 \cdot 3$	1	Y	Y	N
16#7	$2^4 \cdot 3$	2	Y	Y	N
16#8	$2^2 \cdot 3$	2	N	Y	N
32#8	$2^8 \cdot 3$	4	N	Y	Y
32#9	$2^8 \cdot 3^2$	3	Y	Y	N
32#10	$2^6 \cdot 3$	3	N	Y	N
32#15	$2^5 \cdot 3$	2	Y	Y	N
32#17	$2^3 \cdot 3$	2	N	Y	N
32#18	$2^5 \cdot 3$	3	Y	Y	N
32#33	$2^4 \cdot 3$	4	N	N	Y
32#34	$2^6 \cdot 3$	3	N	Y	Y
32#41	$2^3 \cdot 3$	3	N	N	Y
32#42	$2^3 \cdot 3^2$	3	N	Y	Y
32#43	$2^3 \cdot 3 \cdot 5$	2	N	Y	N
64#12	$2^{13} \cdot 3 \cdot 7$	5	N	Y	Y
64#13	$2^{13} \cdot 3^2 \cdot 7$	4	Y	Y	N
64#14	$2^{11} \cdot 3^2$	4	N	Y	N
64#15	$2^{12} \cdot 3$	5	N	N	Y
64#16	$2^{12} \cdot 3$	4	Y	Y	N
64#17	$2^9 \cdot 3$	4	N	N	N

Table 8.1: Groups of order dividing 64 (cont.)

G	$\|\mathrm{Out}(G)\|$	$\mathrm{r}_2(G)$	p-central group	$H^*(G)$ C-M	$V_{\max} \in \mathrm{SCN}(G)$
64#19	$2^{10} \cdot 3$	3	Y	Y	N
64#21	$2^{7} \cdot 3$	3	N	Y	N
64#22	$2^{10} \cdot 3$	4	Y	Y	N
64#27	$2^{8} \cdot 3$	3	N	Y	N
64#30	$2^{10} \cdot 3$	3	Y	Y	N
64#35	$2^{6} \cdot 3$	2	Y	Y	N
64#36	$2^{4} \cdot 3$	2	N	Y	N
64#43	$2^{9} \cdot 3$	4	N	Y	N
64#44	$2^{8} \cdot 3$	4	N	N	N
64#45	$2^{9} \cdot 3$	3	Y	Y	N
64#68	$2^{9} \cdot 3$	5	N	N	Y
64#69	$2^{11} \cdot 3$	4	N	Y	Y
64#76	$2^{8} \cdot 3$	4	N	N	Y
64#81	$2^{9} \cdot 3$	5	N	N	Y
64#82	$2^{9} \cdot 3$	3	Y	Y	N
64#93	$2^{8} \cdot 3$	3	Y	Y	N
64#103	$2^{8} \cdot 3^{2}$	4	N	Y	Y
64#104	$2^{8} \cdot 3 \cdot 5$	3	N	Y	N
64#105	$2^{5} \cdot 3^{2} \cdot 5$	3	N	Y	N
64#107	$2^{6} \cdot 3$	3	N	Y	N
64#108	$2^{8} \cdot 3$	3	N	N	N
64#109	$2^{5} \cdot 3$	3	N	N	N
64#144	$2^{7} \cdot 3$	4	N	Y	Y
64#145	$2^{7} \cdot 3$	3	Y	Y	N
64#147	$2^{7} \cdot 3$	4	N	N	Y
64#148	$2^{7} \cdot 3$	4	N	N	Y

Table 8.1: Groups of order dividing 64 (cont.)

G	$\lvert\mathrm{Out}(G)\rvert$	$\mathrm{r}_2(G)$	p-central group	$H^*(G)$ C-M	$V_{\max} \in \mathrm{SCN}(G)$
64#150	$2^6 \cdot 3$	4	N	N	Y
64#153	$2^6 \cdot 3 \cdot 7$	3	Y	Y	N
64#155	$2^6 \cdot 3$	3	N	Y	N
64#156	$2^7 \cdot 3^2$	2	Y	Y	N
64#158	$2^6 \cdot 3$	3	N	Y	N
64#159	$2^5 \cdot 3$	3	N	Y	N
64#162	$2^6 \cdot 3$	2	Y	Y	N
64#173	$2^7 \cdot 3$	4	N	N	Y
64#181	$2^5 \cdot 3$	3	N	N	N
64#183	$2^6 \cdot 3^2$	4	N	N	Y
64#184	$2^5 \cdot 3$	4	N	N	Y
64#187	$2^6 \cdot 3 \cdot 5$	2	Y	Y	N
64#241	$2^3 \cdot 3$	3	N	Y	N
64#242	$2^2 \cdot 3$	3	N	N	N
64#243	$2^3 \cdot 3$	2	N	Y	N
64#259	$2^2 \cdot 3$	4	N	N	Y
64#260	$2^2 \cdot 3$	3	N	N	Y

An inspection of this table reveals that one of the first three criteria of Theorem 8.3 apply to all but six groups. We will soon show that the question can be resolved for five of these remaining six, and to do so we will apply the fourth criteria of the theorem. Before this explanation, we need some notation and a lemma.

Notation If V is a subgroup of G, we denote the subgroup of $\mathrm{Aut}(G)$ which fixes each element of V by $\mathrm{Fix}(\mathrm{Aut}(G), V)$.

Lemma 8.4 *Let G be a 2-group. With the notation defined above,* $\mathrm{Fix}(\mathrm{Aut}(G), V)$ *is a 2-group if and only if* $\mathrm{Out}(G)_V$ *is a 2-group.*

PROOF: Recall from Section 4.1 that $\mathrm{Out}(G)_V$ is defined to be a certain stabilizer of the action of $\mathrm{Out}(G)$ on $\mathrm{Rep}(W, G)$, where W is an elementary abelian p-group. Specifically, if α_V is an element of $\mathrm{Hom}(W, G)$ with image V, then $\mathrm{Out}(G)_V$ is the stabilizer of $[\alpha_V]$ in $\mathrm{Rep}(W, G)$.

Define $\psi : \mathrm{Fix}(\mathrm{Aut}(G), V) \to \mathrm{Out}(G)_V$ as the map which sends $f \mapsto [f]$. The fact that $\psi(f)$ is contained in $\mathrm{Out}(G)_V$ is straightforward. Our central claim is that ψ is an epimorphism, so let $[f]$ be an element of $\mathrm{Out}(G)_V$. Since $[f] \cdot [\alpha_V] = [\alpha_V]$, there exists an element $g \in G$ such that $f(v) = c_g(v)$ for all $v \in V$. In other words, $c_{g^{-1}} \circ f \in \mathrm{Fix}(\mathrm{Aut}(G), V)$. Then $\psi(c_{g^{-1}} \circ f) = [f]$, meaning ψ is onto.

Since these are finite groups, we have

$$|\,\mathrm{Fix}(\mathrm{Aut}(G), V)| = |\mathrm{Out}(G)_V| \cdot |\ker \psi|.$$

However, since the kernel of ψ is a subgroup of $\mathrm{Inn}(G)$, $\ker \psi$ is a 2-group. This proves the proposition. □

Remark 8.5 Lemma 8.4 holds for p-groups when p is an odd prime as well.

With this lemma in hand, we can use the GAP[1] computer program to determine whether or not $\mathrm{Out}(G)_V$ is a 2-group. In the following table, $V_{\max}$ is an elementary

[1]The author would like to thank the developers of this computer program for making it available to the mathematical community free of charge.

abelian subgroup of rank $\mathrm{r}_p(G)$. (For some of these groups there were several such maximal subgroups, but the size of $\mathrm{Fix}(\mathrm{Aut}(G), V_{\max})$ was identical in each case.)

Table 8.2: Calculations related to the size of $\mathrm{Out}(G)_V$

G	$\lvert \mathrm{Fix}(\mathrm{Aut}(G), V_{\max}) \rvert$
64#17	32
64#44	16
64#108	1536
64#109	64
64#181	64
64#242	8

We conclude that the only outstanding group is 64#108.

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List of Symbols

Index of Terminology